THE HEAT EQUATION UNDER THE RICCI FLOW

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This paper has two main results. The first deals with determining gradient estimates for positive solutions of the heat equation on a manifold whose metric is evolving under the Ricci flow. These are Li-Yau type gradient estimate, and, as an application, Harnack inequalities are given. We consider both the case when the manifold is complete and when it is compact.

The second result consists of an estimate for the fundamental solution of the heat equation on a closed Riemannian manifold of dimension at least 3, evolving under the Ricci flow. The estimate depends on some constants arising from a Sobolev imbedding theorem. Considering the case when the scalar curvature is positive throughout the manifold, at any time, we will obtain, as a corollary, a bound similar to the one known for the fixed metric case.

BIOGRAPHICAL SKETCH

Mihai was born in Craiova, Romania, where he graduated from the "Fraţii Buzeşti" high-school in 2003. In 2006 he obtained a Bachelor of Science in mathematics from Jacobs University Bremen, Germany, which at that time was called International University Bremen. In that same year he started his PhD at Cornell University, where he spent five amazing years.

Dedicated to my mother, Mihaela, and to my father, Constantin Dan.

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CHAPTER 1

INTRODUCTION

1.1 Overview

Since its invention by Richard Hamilton in 1982, the Ricci flow has proven to be a useful analytical tool in attacking and solving geometric problems, from which the most famous are the Poincaré conjecture, the Thurston geometrization conjecture (both proven by G. Perelman in [29] and in [30]), or the recent differential sphere theorem (proven by R. Schoen and S. Brendle in [8]).

Motivated by the fact that the curvatures have a heat-type evolution under the Ricci flow, we investigate the heat equation under the Ricci flow, together with its fundamental solution - the heat kernel.

This paper is structured in four chapters. The first one consists of a short exposition of the main results dealing with the Ricci flow, an introduction of our notation, a presentation of the evolution equations for the curvatures under Ricci flow (which motivate the study of the heat equation under the Ricci flow), the definition of solitons and their significance, a brief description of the seminal results involving Harnack inequalities, and it concludes with some known results related to the heat kernel (which motivate its analysis under Ricci flow).

The second chapter presents gradient estimates under the Ricci flow. There will be two types of estimates, space-only (involving only the spacial deriva-

Material in this thesis includes results from two articles, one was originally published in *M. Bailesteanu, X. Cao, and A. Pulemotov*, **Gradient estimates for the heat equation under the Ricci flow**, J. Funct. Anal., 258(10):3517â3542, 2010, while the other will be published in *M. Bailesteanu*, **Bounds on the Heat Kernel under the Ricci Flow**, Proc. Amer. Math. Soc. (to appear), 2011.

tive) and space-time (involving both the spatial and time derivative). And for each of them we consider two cases: when the manifold is complete (for which we get a local estimate) and when the manifold is compact (for which we get a global estimate). As an application for the space-time estimates, we obtain Harnack inequalities. Most of the chapter is included in [6], which was written in collaboration with Xiaodong Cao and Artem Pulemotov and which contains also the case when the manifold is compact with boundary.

The third chapter consists of determining an upper bound for the heat kernel on a manifold evolving under the Ricci flow. The bound will depend on some constants arising from a Sobolev imbedding theorem. When the scalar curvature is positive throughout the manifold, at any time, we will obtain, as a corollary, a bound similar to the one known for the fixed metric case. The chapter is mostly the article [5].

The final chapter presents some applications of the estimates in the previous two chapter in trying to classify type III solutions of the Ricci flow.

1.2 History of the Ricci flow

One of the fundamental problems in differential geometry is to find canonical metrics on Riemannian manifolds. A way to achieve this is to start with a manifold and, by using a nonlinear heat type flow, evolve it into having one of these canonical metrics. In particular, one can achieve this by using the Ricci flow, which was introduced in 1982 by Richard Hamilton [18]. The Ricci flow is the evolution of the metric under the partial differential equation

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \tag{1.2.1}$$

where $g(x,t)$ represents the Riemannian metric and $\mathrm{Ric}(x,t)$ the corresponding Ricci curvature. In that same paper [18] Hamilton proved the short time existence for any smooth initial metric $g_0(x) = g(x,0)$.

The negative sign shows that positive curvature is contracted, while negative curvature is dilated. This can be noticed immediately if one analyzes manifolds with constant curvature. A sphere, for example, will shrink and collapse to a single point in finite time.

One of Hamilton's first results, using the Ricci flow, was the following theorem:

Theorem 1 (Hamilton, [18]) *Let M be a compact three dimensional manifold, with initial metric of strictly positive Ricci tensor. Then M will admit a metric of constant positive sectional curvature. In particular M is diffeomorphic to a quotient of the three-sphere by a finite group of isometries.*

The proof of this consists in solving the equation 1.2.1 and showing that the renormalized solution (which preserves the volume and which is obtained from the usual one by a homothety and a time change) is defined for all $t \in [0, \infty)$ and converges to a metric of constant sectional curvature. The theorem implies that given a homotopy three-sphere, if one can show that it admits a metric of postivie Ricci curvature, then the Poincaré's conjecture would follow (the conjecture states that every simply connected, closed three-manifold is homeomorphic to the three-sphere).

On the other hand, if one starts with an arbitrary metric, without having any assumptions on the curvature, the Ricci flow solution may develop singularities in finite time. The curvature might become unbounded in some regions, while staying bounded outside of them. Hamilton found a method to deal with these bad regions, namely performing a topological surgery (cutting the neck open and gluing caps on the boundary of the cut) and continuing the Ricci flow. In order to analyze the singularities, a Harnack type inequality was useful, since it allowed comparing the curvature of the evolving manifold at different points and times. As a result, Hamilton found a classification of blow-ups of singularities in dimension three:

Theorem 2 (Hamilton, [21]) *Let $(M, g(t))$ be a solution to the Ricci flow on a compact three-manifold wthere a singularity develops in finite time T. Then either the injectivity radius times the square root of the maximum curvature goes to zero, or else there exists a sequence of dilations of the solution which converges to a quotient by isometries of either S^3, $S^2 \times \mathbb{R}$ or $\Sigma \times \mathbb{R}$, where Σ is the cigar soliton.*

A final step for solving Poincaré's conjecture using Hamilton's program was to prove that all singularities are modelled by self-similar solutions (solitons), i.e. solutions that move by diffeomorphisms and scaling. This was done by Gregory Perelman [29], who introduced new tools to study the Ricci flow and finalized the program. Since the purpose of this paper is not Perelman's proof, we won't describe those tools, but we will mention that they include a new Harnack inequality, the study of the conjugated heat kernel and various entropy functionals.

1.3 Notation

We consider M to be a connected, oriented, smooth, n-dimensional manifold without boundary. If $\{x^i\}_{i=1}^n$ are local coordinatess in a neighborhood U of some point in M, then the vector fields $\{\partial_i = \partial/\partial x^i\}_{i=1}^n$ form a local basis for the tangent bundle TM, while the 1-forms $\{dx^i\}_{i=1}^n$ form a dual basis for T^*M (more generally we denote components of vectors as v^i and components of covectors as v_j). The Riemannian metric, which is a smooth varying inner product on the tangent space (or, equivalently, a positive-definite section of the bundle of symmetric (covariant) 2-tensors), is written in local coordinates as $g = g_{ij}\, dx^i \otimes dx^j$. We will always use the Einstein summation convention, where, for example $x^i y_i := \sum_{i=1}^n x^i y_i$ and $x_i y_i = g^{ij} x_i y_j$, where g^{ij} is the inverse of the Riemannian metric g_{ij}.

The Levi-Civita connection (or the covariant derivative) $\nabla : TM \times C^\infty(TM) \to C^\infty(TM)$ is the unique connection on TM that is compatible with the metric and torsion free:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\,,$$

$$\nabla_x Y - \nabla_Y X = [X, Y]\,.$$

∇_k will stand for the covariant derivative in the $\frac{\partial}{\partial x^k}$ direction, and the connection is determined by the Christoffel symbols:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right).$$

For functions, the notation f_i stands for $\frac{\partial f}{\partial x_i}$, the notation f_{ij} refers to $\nabla_i \nabla_j f$ (taking the covariant derivative twice in the direction of $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$). The subscript t designates the differentiation in $t \in [0, T]$.

The Riemann curvature tensor is a (3,1) tensor defined as

$$\mathrm{Rm}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and, in a local coordinate frame, its coefficients are given by

$$\mathrm{Rm}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = R^l_{ijk}\frac{\partial}{\partial x_l}.$$

These coefficients can be calculated from the Christoffel symbols as follows:

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}\,.$$

Lowering the upper index we get a (4,0) tensor $R_{ijkl} = g_{lm}R^m_{ijk}$, which is anti-symmetric in (i, j) and (k, l) and symmetric in the interchange of these pairs.

If $P \subset T_x M$ is a 2-plane, then the sectional curvature of P is defined by

$$K(P) = g(\mathrm{Rm}(e_1, e_2)e_2, e_1)\,,$$

for $\{e_1, e_2\}$ being an orthonormal basis of P.

The Ricci tensor Ric is the trace of the Riemann curvature tensor, being given in local coordinates by

$$R_{jk} = R^i_{ijk}\,,$$

while the scalar curvature is the trace of the Ricci tensor

$$R = g^{ij}R_{ij}\,.$$

If we consider $\{e_1, ..., e_n\}$ an orthonormal basis for the tangent space, then we can write the two curvatures as

$$\mathrm{Ric}(X, Y) := \sum_{i=1}^{n} g(\mathrm{Rm}(X, e_i)e_i, Y)\,,$$

$$R := \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)\,.$$

We conclude this section with the Bianchi identities, which we may use implicitly in our calculations:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0\,,$$
$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$$

1.4 Evolution equations under Ricci flow

If given $T > 0$, $(M, g(x,t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.2.1), then the curvatures will also change in time, together with the connection, the volume element and the Laplacian. Let's just mention that we will denote with Δ the Laplacian given by $g(x,t)$ (even though it is time dependent).

The following results were proven by Hamilton [18].

Theorem 3 *Under the Ricci flow, the evolution equations of the curvatures are the following:*

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{iklj} - B_{iljk}) \tag{1.4.1}$$
$$- (R_{ip}R_{pjkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp})\,,$$
$$\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + 2R_{kijl}R_{kl} - 2R_{ik}R_{jk}\,, \tag{1.4.2}$$
$$\frac{\partial}{\partial t}R = \Delta R + 2|\operatorname{Ric}|^2\,, \tag{1.4.3}$$

where $B_{ijkl} = -R_{pijq}R_{qlkp}$.

An immediate observation one can make is that, since $|\operatorname{Ric}|^2 \geq \frac{1}{n}R^2$, by the maximum principle,

$$R(x,t) \geq \frac{n}{n(\inf_{t=0} R)^{-1} - 2t}$$

for all $x \in M$ and $t \geq 0$ (we will use this fact later, in our computations). From this it also follows, for M closed, that if at the initial time M has positive scalar curvature, the singularity, if it occurs, will do so in finite time.

Another observation is that the curvatures evolve under a non-linear heat equation. This gives a strong motivation to study the heat equation under Ricci flow, more precisely one would want to obtain Harnack inequalities of these solutions.

As the metric is changing, so will the connections, the Laplacian and the volume element. Here are their evolutions (for a proof see [15]):

Theorem 4

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right), \tag{1.4.4}$$

$$\frac{\partial}{\partial t}\Delta = 2R_{ij}\nabla_i\nabla_j, \tag{1.4.5}$$

$$\frac{\partial}{\partial t}dv = -R\,dv. \tag{1.4.6}$$

The last formula shows, for example, that a sphere (which has constant positive scalar curvature) will shrink to a point under the Ricci flow.

1.5 Ricci solitons

An important concept in the study of evolution equations are the solutions that change under a one-parameter group of symmetries. These are called "self-similar" solutions or "solitons". The symmetry group for the Ricci flow contains all the diffeomorphisms (modulo scalings), hence the solitons will be solutions which move by diffeomorphisms and scaling.

More precisely, $g(t)$ is a soliton if it is a pull-back of the initial metric $g(0)$, i.e.

$$g(t) = \alpha(t)\Psi_t^*(g(0))\,,$$

where $\Psi_t : M \to M$ is a diffeomorphism for each t, with $\Psi_0 = Id$ and $\alpha(t)$ is a real-valued function. If $\alpha(t) < 1$, $\alpha(t) = 1$ and $\alpha(t) > 1$ then we call the soliton "shrinking", "steady" and "expanding" respectively.

Moreover, if Ψ is generated by a vector field X ($\partial_t \Psi_t = X \circ \Psi_t$), which, in turn, is the gradient of a function f ($X_i = \nabla_i f$), then $g(t)$ is called a "gradient soliton".

Examples of solitons include the "cigar" soliton and the Bryant soliton. The cigar soliton Σ was introduced by Hamilton and it is the complete Riemann surface $\mathbb{R}^2$ with the initial metric $g_\Sigma(0) = \frac{dx^2+dy^2}{1+x^2+y^2}$ which gives the Ricci flow solution $g_\Sigma(t) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}$. In physics literature it is known as the Witten's black hole. The Bryant soliton is, up to homothety the unique rotationally symmetric complete, steady gradient Ricci soliton metric on $\mathbb{R}^n$ (for $n \geq 3$) with positive curvature operator.

Gradient solitons are connected to the conjugate heat equation

$$\partial_t u + \Delta u - Ru = 0\,, \tag{1.5.1}$$

where R is the scalar curvature at the point (x,t). For example, in case of a gradient steady soliton, if one denotes with $u := e^{-f}$, where f is the potential function that generates the vector field X, then u satisfies (1.5.1). This equation is not solvable in positive time, but one can still obtain some information about its solution. This is done as follows: consider a solution $g(t)$ to the Ricci flow, for some $t \in [0,T]$, fix a final time data $u(T) = u_0$ and, with a new variable $\tau = T - t$, analyze the following equation $\partial_\tau u(\tau) - \Delta u(\tau) + Ru(\tau) = 0$. The latter is parabolic and the usual theory can be utilized.

As we have seen, for example, in the overview section, in theorem 2, solitons are important in modeling the solutions of the Ricci flow near singularities.

1.6 Differential Harnack inequalities

The classical Harnack inequality states that a non-negative smooth function $u : M \times [0, T] \to [0, \infty)$, which is a solution to the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x, t) = 0, \qquad x \in M, \ t \in [0, T]. \tag{1.6.1}$$

satisfies the following

$$\sup_{x \in M} u(x, t_1) \leq C \inf_{x \in M} u(x, t_2)$$

for $0 < t_1 < t_2 \leq T$, $T > 0$ and C depending on t_1, t_2 and the geometry of the manifold.

From this it follows that if $u(x_0, t_0) = 0$ for some $x_0 \in M$ and $t_0 > 0$, then $u \equiv 0$ on $M \times [0, t_0]$. Moreover, if $u(x_0, 0) > 0$ for some $x_0 \in M$, then at any time $t > 0$ one has that $u(x, t) > 0$ for all $x \in M$. This result has some disadvantages, since it only gives qualitative information, and the geometric dependency on C is complicated.

In their seminal paper [25] from 1986, Peter Li and S.T. Yau found pointwise gradient estimates, which were later called "Li-Yau differential Harnack inequalities", from which one cand find classical Harnack inequalities by integrating the estimates on a space-time path. Being pointwise, these inequalities can be used to analyse the function locally. First they proved a pointwise global result, for compact manifolds:

Theorem 5 (Li-Yau, [25]) *Let M be a compact n-dimensional Riemannian manifold, possibly with boundary and with Ricci curvature* $\mathrm{Ric}(M) \geq 0$. *Let $u : M \times [0, \infty) \to [0, \infty)$ be a non-negative solution to the heat equation (1.6.1). If the manifold has boundary, assume it is convex and that $u(x, t)$ satisfies the Neumann boundary condition $\partial u(x, t)/\partial \nu = 0$ on the boundary for any time $t \in (0, \infty)$, where $\partial/\partial \nu$ is the outer normal vector on the boundary. Then u satisfies the following gradient estimate on $M \times (0, \infty)$*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}. \tag{1.6.2}$$

The inequality becomes equality if M is the Euclidean space and u is the heat kernel, i.e. $u(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$.

They next proved a local result, in a more general setting.

Theorem 6 (Li-Yau, [25]) *Let M be a complete n-dimensional Riemannian manifold, with Ricci curvature* $\mathrm{Ric}(M) \geq -k$, *for $k \geq 0$ and let $B_{2\rho}$ be a geodesic ball of radius 2ρ centered at some point in M. If $u : B_{2\rho} \times [0, \infty) \to [0, \infty)$ is a non-negative solution to the heat equation (1.6.1), then for any $\alpha > 1$, the following gradient estimate holds in $B_\rho \times (0, \infty)$*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq C\alpha^2 \left(\frac{\alpha^2}{\rho^2(\alpha - 1)} + \sqrt{k}\rho \right) + \frac{nk\alpha^2}{2(\alpha - 1)} + \frac{n\alpha^2}{2t} \tag{1.6.3}$$

where C depends only on the dimension of M.

Let us mention that this local inequality can be used to prove the global one.

By integrating over space-time paths, we obtain the following classical Harnack inequality:

Theorem 7 (Li-Yau, [25]) *Let M be a complete, compact or noncompact, n-dimensional manifold, with* $\mathrm{Ric}(M) \geq -k$*, for* $k \geq 0$*. If* $u : M \times [0, \infty) \to [0, \infty)$ *is a non-negative solution to the heat equation (1.6.1), then for any* $\alpha > 1$, $x_1, x_2 \in M$ *and* $0 < t_1 < t_2 < \infty$ *the following inequality holds:*

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{n\alpha/2} e^{\frac{\alpha^2 d^2(x_1, x_2)}{4(t_2 - t_1)} + \frac{nk\alpha}{2(\alpha - 1)}(t_2 - t_1)} . \tag{1.6.4}$$

In particular, if $k = 0$ *the above becomes:*

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{n/2} e^{\frac{d^2(x_1, x_2)}{4(t_2 - t_1)}} . \tag{1.6.5}$$

Proving all these involves using the maximum principle for parabolic equations.

A few years later, in 1993, by using a maximum principle for tensors that he himself invented, Hamilton proved a matrix version of the Li-Yau differential Harnack inequality:

Theorem 8 (Hamilton, [19]) *Let M be a compact n-dimensional manifold, which is Ricci parallel and has weakly postive sectional curvature* ($\nabla \mathrm{Ric} = 0$ *and* $R_{ijkl} v_i w_j v_k w_l \geq 0$ *for all v, w vector fields). If* $u : M \times [0, T] \to (0, \infty)$ *is a non-negative solution to the heat equation (1.6.1), then the following inequality holds:*

$$\nabla_i \nabla_j u - \frac{\nabla_i u \nabla_j u}{u} + \frac{u}{2t} g_{ij} \geq 0 . \tag{1.6.6}$$

In particular, with restrictions on the Ricci curvature, by taking the trace the result from theorem 5 follows.

Just as in Li-Yau's case, the expression (1.6.6) becomes equality if u is the Euclidean heat kernel. This is very important, because the heat kernel can be seen as the self-similar solution of the heat equation.

In summary, from studying the heat equation on manifolds, one can notice that in order to compare solutions at different points and times, one needs inequalities that become equalities on self-similar solutions.

Hence, for the Ricci flow, if one wants to compare the curvatures at different points and times (and, as we have seen, curvatures have a heat-type evolution), one needs to study the self-similar solutions, which are the Ricci solitons. Motivated by this, Hamilton looked for curvature expressions that vanish on expanding Ricci solitons and searched for linear combinations of these expressions, and in [21] managed to find a matrix and a trace Harnack inequality for the Ricci flow.

Theorem 9 (Hamilton, [21]) *Suppose (M, g) is a complete solution to the Ricci flow (1.2.1) for $t \in [0, T]$. Suppose also that g_{ij} has weakly curvature positive curvature operator. Define*

$$M_{ij} = \Delta R_{ij} - \frac{1}{2}\nabla_i \nabla_j R + 2R_{ikjl}R_{kl} - R_{ik}R_{jk} + \frac{1}{2t}R_{ij}\,.$$

Then for any vectors V, W and $t > 0$ we have:

$$H(V, W) := 2M_{ij}W_iW_j - 4(\nabla_i R_{jk})(V_iW_j - V_jW_i)W_k + 2R_{ijkl}V_iW_jV_kW_l \geq 0\,, \qquad (1.6.7)$$

$$H(V) := \frac{\partial R}{\partial t} + \frac{R}{t} - 2\nabla_i RV_i + 2R_{ij}V_iV_j \geq 0\,. \qquad (1.6.8)$$

Using this matrix Harnack, Hamilton managed to prove the classification of blow-ups from theorem 2.

Further, G. Perelman found in [29] a Harnack inequality for the conjugate heat equation (1.5.1). More precisely, he showed that if u satisfies the conjugate heat equation, then f, which is defined as $u = (4\pi(T - t))^{-n/2}e^{-f}$, satisfies

$$-\frac{d}{dt}f(\gamma(t),t) \leq \frac{1}{2}(R(\gamma(t),t) + |\dot{\gamma}|^2) - \frac{1}{2(T-t)f(\gamma(t),t)}\,,$$

where T is the final time for the Ricci flow and $\gamma(t)$ is any smooth curve in M. This inequality was later used in the analysis of the singularities.

1.7 Heat kernel under Ricci flow

The heat kernel represents the fundamental solution to the heat equation (1.6.1) on a particular domain with appropriate boundary conditions. On $\mathbb{R}^n$ it has an exact expression:

$$G(x,t;y,s) = \begin{cases} \frac{1}{[4\pi(t-s)]^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} & t > s\,, \\ 0 & t < s\,, \end{cases}$$

but on a general complete (non-compact) manifold it is defined as being, for each (y,s), the minimal solution $u(x,t) = G(x,t;y,s)$ of the system:

$$\begin{cases} (\partial_t - \Delta)u(x,t) = 0\,, \\ u(0,x) = \delta_y(x)\,, \end{cases}$$

where δ_y is the Dirac-delta function.

Alternatively $G(x,t;y,s)$ is a smooth function, $(x,y) \in M \times M$, $s < t$ such that $\Delta(G) - G_t = 0$ in $(x,t) \neq (y,s)$ and for each $y \in M$ $\lim\limits_{t\to s} G(\cdot,t;y,s) = \delta_y$ (the Dirac delta function). It satisfies the following properties:

- positivity: $G(x,t;y,s) > 0$ for all $x, y \in M$, $s < t$,
- $\int\limits_M G(x,t;y,s)\, d\mu(y,s) = 1$ for all $x \in M$, $s < t$,

- semigroup property: $G(x,t;y,s) = \int_M G(x,t;z,\tau)G(z,\tau;y,s)\ d\mu(z,\tau)$ for $\tau \in (s,t)$.

If M is compact, then it is also the unique function satisfying these properties.

One would like to be able to estimate $G(x,t;y,s)$ as well as possible, depending on the constraints of the problem (the geometry of the manifold, the boundary, if it has any etc). Ideally, the estimates should be Gaussian (because in the Euclidean case $G(x,t;y,s)$ is itself Gaussian). If it is not possible to obtain Gaussian bounds, then one would want to get on-diagonal bounds ($x = y$), i.e. $G(x,t;x,s) \approx [4\pi(t-s)]^{-n/2}$.

Using their gradient estimate (1.6.4), Li and Yau obtained in [25] Gaussian bounds for the case when M is complete compact or compact with convex boundary (with Neumann boundary conditions):

$$G(x,t;y,s) \leq C(\delta,n)V_x^{-1/2}(\sqrt{t-s})V_y^{-1/2}(\sqrt{t-s}) \cdot e^{\left[-\frac{\mathrm{dist}^2(x,y)}{(4+\delta)(t-s)}+C_1\delta k(t-s)\right]},$$

where $V_x(\rho)$ is the volume of the ball of radius ρ centered at x, $C(\delta,n) \to \infty$ as $\delta \to 0$, C_1 depends only on n and $\mathrm{Ric} \geq -k$, $k \geq 0$. When $\mathrm{Ric} > 0$ they obtained a lower bound too (also Gaussian):

$$G(x,t;y,s) \geq C^{-1}(\delta,n)V_x^{-1/2}(\sqrt{t-s})V_y^{-1/2}(\sqrt{t-s}) \cdot e^{\left[-\frac{\mathrm{dist}^2(x,y)}{(4+\delta)t}\right]}.$$

A few years later, Jiaping Wang generalized in [34] Li-Yau's heat kernel estimates, to the case when the boundary of the manifold is non-convex. More precisely, the boundary satisfied the "interior rolling ball" condition (for each point on the boundary, there is a geodesic ball centered at some point in the manifold and contained entirely in the manifold such that the point is inside the ball). The author obtained some heat kernel bounds similar in nature to the ones proved by Li-Yau and used them to estimate the Neumann Sobolev constant of M. This constant comes from the following type of Sobolev inequality

$$\inf_{k\in\mathbb{R}}\left(\int_M |f-k|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C(S)\int_M |\nabla f|^2\,,$$

for $f \in C^\infty(M)$. The heat kernel bounds provided a way to estimate $C(S)$ in terms of the geometry of M. In fact, he managed to prove that

$$G(x,t;y,s) \leq C(S)(t-s)^{-n/2}\,,$$

and from this he obtained a bound for $C(S)$ which depends on the dimension of M and the bounds on the curvature, second fundamental form and the boundary conditions.

In the case when the metric changes in time, the first results were obtained by C. Guenther [16]. She proved the existence of a fundamental solution for the linear parabolic operator $L(u) = (\Delta - \partial_t - h)u$ (for $h(x,t)$ a smooth function) on a compact manifold, with time changing metric. Moreover, she also showed that the uniqueness, positivity and the semigroup property hold. In particular, if $g(t)$ is a solution to the Ricci flow and $h(x,t) = 0$, then one has the existence and the properties of the heat kernel under the Ricci flow.

Let us mention that under the Ricci flow, the following holds: G satisfies the heat equation in the (x,t) coordinates

$$\Delta_x G(x,t;y,s) - \partial_t G(x,t;y,s) = 0\,,$$

whereas in the (y,s) it satisfies the conjugate heat equation

$$\Delta_y G(x,t;y,s) + \partial_s G(x,t;y,s) - R(y,s)G(x,t;y,s) = 0\,,$$

here $R(y,s)$ is the scalar curvature, measured with respect to the metric $g(s)$. As a result, one may try to find bounds for the heat kernel either by inspecting the heat equation or the conjugate heat equation. In some cases (for example, for ancient solutions, i.e. solutions that exist for $t \in (-\infty, T)$ for $T > 0$), the latter is easier.

For example, in [35] Q. Zhang considered the conjugate heat equation, after a time reversal (hence it becomes parabolic):

$$\Delta u - \frac{\partial}{\partial t} - Ru = 0\,,$$

and the metric evolving under the (now forward in the time reversal) Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = 2\,\mathrm{Ric}(x,t)\,,$$

on a complete Riemannian manifold. He obtained un upper bound on the fundamental solution using the Nash method, assuming $Ric(g(t)) \geq -k$ and the injectivity radius $i > 0$. The bound depends on the best constants in the Sobolev imbedding theorem.

Most recently, in [11] Q. Zhang and X. Cao, by means of bounds on the heat kernel, proved the following classification result:

Theorem 10 *Let $(M, g(t))$, $t \in (-\infty, 0]$ be a non-flat, type I κ-solution to the Ricci flow. Then there is a sequence of points $\{q_k\} \subset M$, a sequence of times $t_k \to -\infty$ and a sequence of rescaled metrics*

$$g_k(x,s) = |t_k|^{-1} g(x, t_k - s|t_k|)$$

around q_k such that (M, g_k, q_k) converge to a non-flat gradient shrinking Ricci soliton in C^∞_{loc} topology.

We will define what type I κ-solution means in the last chapter. What is important to note is that their proof was based essentially on proving that

$$\frac{a}{t^{n/2}} \leq G(x_0, 0; x_0, -t) \leq \frac{b}{t^{n/2}}\,,$$

for some constants a, b. This, in turn, followed from a Perelman type Harnack inequality for the conjugate heat equation $\Delta u + u_t - Ru = 0$, which was discussed above.

From the above, one can observe a motivation to study the heat kernel under the Ricci flow and to obtain bounds for it in settings as general as possible.

Finally, we conclude with a list of good books, where one can find a more detailed exposition of the Ricci flow and its applications: [27], [32], [15], [24], [12], [13], [26] and [36].

CHAPTER 2

GRADIENT ESTIMATES OF THE HEAT EQUATION UNDER THE RICCI FLOW

2.1 Introduction

This chapter deals with a manifold M evolving under the Ricci flow and with positive solutions to the heat equation on M. We establish a series of gradient estimates for such solutions including several Li-Yau-type inequalities. We study the cases when M is a complete manifold and when M is compact. Our results contain estimates of both local and global nature.

Suppose M is a manifold without boundary. Let $(M, g(x,t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \qquad x \in M,\ t \in [0,T]. \tag{2.1.1}$$

We assume its curvature remains uniformly bounded for all $t \in [0,T]$. Consider a positive function $u(x,t)$ defined on $M \times [0,T]$. We assume $u(x,t)$ solves to the equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad x \in M,\ t \in [0,T]. \tag{2.1.2}$$

Problem (2.1.1) combined with (2.1.2) admits a simple interpretation in terms of the process of heat conduction. More specifically, one may think of the manifold M with the initial metric $g(x,0)$ as an object having the temperature distribution $u(x,0)$. Suppose we let M evolve under the Ricci flow and simultaneously let the heat spread on M. Then the solution $u(x,t)$ will represent the temperature of M at the point x at time t.

The study of system (2.1.1)–(2.1.2) arose from R. Hamilton's paper [17]. The original idea in [17] was to investigate the Ricci flow combined with the heat flow of harmonic maps. The system we examine in this chapter may be viewed as a special case. We point out, without a deeper explanation, that looking at the two evolutions together leads to interesting simplifications in the analysis.

After its conception in [17], the study of (2.1.1)–(2.1.2) was pursued in [16, 28, 35, 1, 10]. A large amount of work was done to understand several problems that are similar to (2.1.1)–(2.1.2) in one way or another. The list of relevant references includes but is not limited to [35, 9, 10] and [13, Chapter 16]. For instance, there are substantial results concerning the Ricci flow combined with the conjugate heat equation. The connection of this problem to (2.1.1)–(2.1.2) is beyond superficial. Q. Zhang used a gradient estimate for (2.1.1)–(2.1.2) to prove a Gaussian bound for the conjugate heat equation in [35]. The results of the present paper may have analogous applications.

As we have seen in the introduction chapter, the scalar curvature of a surface which evolves under the Ricci flow satisfies the heat equation with a potential on that surface. The study of the system (2.1.1)–(2.1.2) would be hence a good starting point in trying to better understand the behaviour of curvatures. Moreover, L. Ni's work [28] offers yet another way to use the Ricci flow combined with the heat equation to study the evolution of $g(x, t)$.

Section 2.3 discusses space-only gradient estimates for system (2.1.1)–(2.1.2) for a bounded function u. A local space-only gradient estimate for solutions of (2.2.2) was originally proved in the paper [33] in the situation where $g(x, t)$ did not depend on $t \in [0, T]$ and (2.2.1) was not in the picture. It was further generalized in [35] to hold in the case of the backward Ricci flow combined with the heat equation. Our Theorem 12 constitutes a version of this result for $u(x, t)$.

A global space-only gradient estimate for solutions of (2.2.2) was originally established in [19] with $g(x, t)$ independent of $t \in [0, T]$ and (2.2.1) not assumed. It is now known to hold in the cases of both the backward Ricci flow and the Ricci flow combined with the heat equation; see [35, 9, 10]. We restate it in Theorem 14 for the completeness of our exposition. New versions of R. Hamilton's result were proposed in [33, 35, 9, 10]. Theorem 12 states a local space-only gradient estimate for (2.1.1)–(2.1.2).

Section 2.4 deals with space-time gradient estimates for (2.1.1)–(2.1.2). The results resemble the Li-Yau inequalities from the paper [25], which we have presented in theorems 6 and 5; see also [32, Chapter IV]. Recalling to the reader, the solution $u(x, t)$ of equation (2.1.2) satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}, \qquad x \in M, \ t \in (0, T],$$

if M is a closed manifold with nonnegative Ricci curvature, the metric $g(x, t)$ does not depend on t. This result goes back to [3, 25] and constitutes the simplest Li-Yau inequality. As mentioned in the introduction, It opened new possibilities for the comparison of the values of solutions of (2.1.2) at different points and led to important Gaussian bounds in heat kernel analysis. Other variants of the above estimate exist in the literature (see, for example, [14, 7]. R. Hamilton proved a matrix version of it in [19]. For the Ricci flow, the Li-Yau-type inequality became one of the central tools in classifying ancient solutions to the flow as detailed in [15, Chapter 9]. Analogous results played a significant part in the study of Kähler manifolds; see [12, Chapter 2].

Theorems 16 and 17 establish space-time gradient estimates for (2.1.1)–

(2.1.2). As an application, we obtain two Harnack inequalities for (2.1.1)–(2.1.2). They help compare the values of a solution at different points. We hope that the techniques in Section 2.4 will lead to the discovery of new informative Li-Yau-type inequalities related to the Ricci flow and other geometric flows.

2.2 Setup

Suppose M is a connected, oriented, smooth, n-dimensional manifold without boundary. Some of the results concern the case where M is compact. Given $T > 0$, assume $(M, g(x,t))_{t\in[0,T]}$ is a complete solution to the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \qquad x \in M,\ t \in [0,T]. \tag{2.2.1}$$

Suppose a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfies the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad x \in M,\ t \in [0,T]. \tag{2.2.2}$$

Recall that Δ stands for the Laplacian given by $g(x,t)$ (it is time dependent).

The results in this chapter are still valid, with obvious modifications, if the function $u(x,t)$ is defined on $M \times (0,T]$ instead of $M \times [0,T]$. In order to see this, it suffices to replace $u(x,t)$ and $g(x,t)$ with $u(x,t+\epsilon)$ and $g(x,t+\epsilon)$ for a sufficiently small $\epsilon > 0$, apply the corresponding formula, and then let ϵ go to 0. We thus justify, for example, the application of the theorems in Subsection 2.4 to heat-kernel-type functions.

Let's introduce another piece of notation: Let us fix $x_0 \in M$ and $\rho > 0$. We write $\mathrm{dist}(\chi, x_0, t)$ for the distance between $\chi \in M$ and x_0 with respect to the metric $g(x,t)$. The notation $B_{\rho,T}$ stands for the set $\{(\chi,t) \in M \times [0,T] \,|\, \mathrm{dist}(\chi, x_0, t) < \rho\}$. We point out that Theorems 12 and 16 still hold if $u(x,t)$ is defined on $B_{\rho,T}$ instead of $M \times [0,T]$ and satisfies the heat equation in $B_{\rho,T}$.

The proofs in this chapter will often involve local computations. Therefore, we assume a coordinate system $\{x_1, \ldots, x_n\}$ is fixed in a neighborhood of every point $x \in M$. Let's recall that R_{ij} refers to the corresponding components of the Ricci tensor. In order to facilitate the computations, we often implicitly assume that $\{x_1, \ldots, x_n\}$ are normal coordinates at $x \in M$ with respect to the appropriate metric. We use the standard notation: for a real-valued function f on the manifold M, $f_i := \frac{\partial f}{\partial x_i}$, f_{ij} refers to the Hessian of f applied to $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$, and f_{ijk} is the third covariant derivative applied to $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_j}$, and $\frac{\partial}{\partial x_k}$. The subscript t designates the differentiation in $t \in [0,T]$.

The proofs of Theorems 12 and 16 will make use of a cut-off function on $B_{\rho,T}$. The construction of this function will rely on the basic analytical result stated in the following lemma. This result is well-known. For example, it was previously used in the proofs of Theorems 2.3 and 3.1 in [35]; see also [32, Chapter IV] and [33].

Lemma 11 *Given $\tau \in (0,T]$, there exists a smooth function $\bar{\Psi} : [0,\infty) \times [0,T] \to \mathbb{R}$ satisfying the following requirements:*

1. *The support of $\bar{\Psi}(r,t)$ is a subset of $[0,\rho] \times [0,T]$, and $0 \le \bar{\Psi}(r,t) \le 1$ in $[0,\rho] \times [0,T]$.*
2. *The equalities $\bar{\Psi}(r,t) = 1$ and $\frac{\partial \bar{\Psi}}{\partial r}(r,t) = 0$ hold in $\left[0, \frac{\rho}{2}\right] \times [\tau, T]$ and $\left[0, \frac{\rho}{2}\right] \times [0,T]$, respectively.*
3. *The estimate $\left|\frac{\partial \bar{\Psi}}{\partial t}\right| \le \frac{\bar{C}\bar{\Psi}^{\frac{1}{2}}}{\tau}$ is satisfied on $[0,\infty) \times [0,T]$ for some $\bar{C} > 0$, and $\bar{\Psi}(r,0) = 0$ for all $r \in [0,\infty)$.*

4. *The inequalities* $-\frac{C_a\bar{\Psi}^a}{\rho} \leq \frac{\partial\bar{\Psi}}{\partial r} \leq 0$ *and* $\left|\frac{\partial^2\bar{\Psi}}{\partial r^2}\right| \leq \frac{C_a\bar{\Psi}^a}{\rho^2}$ *hold on* $[0,\infty)\times[0,T]$ *for every* $a \in (0,1)$ *with some constant* C_a *dependent on* a.

2.3 Space-only gradient estimates

Let us begin by stating the local space-only gradient estimate.

Theorem 12 *Suppose* $(M, g(x,t))_{t\in[0,T]}$ *is a complete solution to the Ricci flow* (2.2.1). *Assume that* $|\operatorname{Ric}(x,t)| \leq k$ *for some* $k > 0$ *and all* $(x,t) \in B_{\rho,T}$. *Suppose* $u : M\times[0,T] \to \mathbb{R}$ *is a smooth positive function solving the heat equation* (2.2.2). *If* $u(x,t) \leq A$ *for some* $A > 0$ *and all* $(x,t) \in B_{\rho,T}$, *then there exists a constant* C *that depends only on the dimension of* M *and satisfies*

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{\rho} + \frac{1}{\sqrt{t}} + \sqrt{k}\right)\left(1 + \log\frac{A}{u}\right) \tag{2.3.1}$$

for all $(x,t) \in B_{\frac{\rho}{2},T}$ *with* $t \neq 0$.

We will now establish a lemma of computational character. It will play an significant part in the proof Theorem 12.

Lemma 13 *Let* $(M, g(x,t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (2.2.1). *Consider a smooth positive function* $u : M\times[0,T] \to \mathbb{R}$ *satisfying the heat equation* (2.2.2). *Assume that* $u(x,t) \leq 1$ *for all* $(x,t) \in B_{\rho,T}$. *Let* $f = \log u$ *and* $w = \frac{|\nabla f|^2}{(1-f)^2}$. *Then the inequality*

$$\left(\Delta - \frac{\partial}{\partial t}\right)w \geq \frac{2f}{1-f}\nabla f\nabla w + 2(1-f)w^2$$

holds in $B_{\rho,T}$.

PROOF. A direct computation demonstrates that

$$\left(\Delta - \frac{\partial}{\partial t}\right) w = \sum_{i,j=1}^{n} \left(\frac{2f_{ij}^2}{(1-f)^2} + 8\frac{f_i f_{ij} f_j}{(1-f)^3} - 4\frac{f_i f_j f_{ij}}{(1-f)^2} \right) + 6\frac{|\nabla f|^4}{(1-f)^4} - 2\frac{|\nabla f|^4}{(1-f)^3}$$

and

$$4\sum_{i,j=1}^{n} \frac{f_i f_{ij} f_j}{(1-f)^3} = 2\frac{\nabla f \nabla w}{(1-f)} - 4\frac{|\nabla f|^4}{(1-f)^4}$$

at every point $(x,t) \in B_{\rho,T}$; cf. [33, 35]. Using these formulas, we conclude that

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) w &= \sum_{i,j=1}^{n} \left(\frac{2f_{ij}^2}{(1-f)^2} + 4\frac{f_i f_{ij} f_j}{(1-f)^3} - 4\frac{f_i f_j f_{ij}}{(1-f)^2} \right) \\
&\quad + 2\frac{|\nabla f|^4}{(1-f)^4} + 2\frac{\nabla f \nabla w}{(1-f)} - 2\frac{|\nabla f|^4}{(1-f)^3} \\
&= 2\sum_{i,j=1}^{n} \left(\frac{f_{ij}}{1-f} + \frac{f_i f_j}{(1-f)^2} \right)^2 \\
&\quad + 2\frac{\nabla f \nabla w}{(1-f)} + 2\frac{|\nabla f|^4}{(1-f)^3} - 2\nabla f \nabla w \\
&\geq \frac{2f}{1-f} \nabla f \nabla w + 2(1-f) w^2
\end{aligned}$$

at $(x,t) \in B_{\rho,T}$.

The preparations required to prove Theorem 12 are now completed. Note that we will also make use of arguments from the paper [35].

PROOF. [Proof of Theorem 12.] Without loss of generality, we can assume $A = 1$. If this is not the case, one should just carry out the proof replacing $u(x,t)$ with $\frac{u(x,t)}{A}$. Let us pick a number $\tau \in (0,T]$ and fix a function $\bar{\Psi}(r,t)$ satisfying the conditions of Lemma 11. We will establish (2.3.1) at (x,τ) for all x such that $\text{dist}(x,x_0,\tau) < \frac{\rho}{2}$. Because τ is chosen arbitrarily, the assertion of the theorem will immediately follow.

Define $\Psi : M \times [0,T] \to \mathbb{R}$ by the formula

$$\Psi(x,t) = \bar{\Psi}(\text{dist}(x,x_0,t),t).$$

It is easy to see that $\Psi(x,t)$ is supported in the closure of $B_{\rho,T}$. This function is smooth at $(x',t') \in M \times [0,T]$ whenever $x' \neq x_0$ and x' is not in the cut locus of x_0 with respect to the metric $g(x,t')$. We will employ the notation $f = \log u$ and $w = \frac{|\nabla f|^2}{(1-f)^2}$ introduced in Lemma 13. It will also be convenient for us to write β instead of $-\frac{2f}{1-f}\nabla f$. Our strategy is to estimate $\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi w)$ and scrutinize the produced formula at a point where Ψw attains its maximum. The desired result will then follow.

We use Lemma 13 to conclude that

$$\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi w) \geq \Psi\left(-\beta\nabla w + 2(1-f)w^2\right) + (\Delta\Psi)w + 2\nabla\Psi\nabla w - \Psi_t w$$

in the portion of $B_{\rho,T}$ where $\Psi(x,t)$ is smooth. This implies

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi w) \geq & -\beta\nabla(\Psi w) + \frac{2}{\Psi}\nabla\Psi\nabla(\Psi w) + 2\Psi(1-f)w^2 \\ & + w\beta\nabla\Psi - 2\frac{|\nabla\Psi|^2}{\Psi}w + (\Delta\Psi)w - \Psi_t w. \end{aligned} \tag{2.3.2}$$

The latter inequality holds in the part of $B_{\rho,T}$ where $\Psi(x,t)$ is smooth and nonzero. Now let (x_1,t_1) be a maximum point for Ψw in the closure of $B_{\rho,T}$. If $(\Psi w)(x_1,t_1)$ is equal to 0, then $(\Psi w)(x,\tau) = w(x,\tau) = 0$ for all $x \in M$ such that $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. This yields $\nabla u(x,\tau) = 0$, and estimate (2.3.1) becomes obvious at (x,τ). Thus, it suffices to consider the case where $(\Psi w)(x_1,t_1) > 0$. In particular, (x_1,t_1) must be in $B_{\rho,T}$, and t_1 must be strictly positive.

A standard argument due to E. Calabi (see, for example, [32, page 21]) enables us to assume that $\Psi(x,t)$ is smooth at (x_1,t_1). Because (x_1,t_1) is a maximum point, the equalities $\Delta(\Psi w)(x_1,t_1) \leq 0$, $\nabla(\Psi w)(x_1,t_1) = 0$, and $(\Psi w)_t(x_1,t_1) \geq 0$ are satisfied. Together with formula (2.3.2), they yield

$$2\Psi(1-f)w^2 \leq -w\beta\nabla\Psi + 2\frac{|\nabla\Psi|^2}{\Psi}w - (\Delta\Psi)w + \Psi_t w \tag{2.3.3}$$

at (x_1, t_1). We will now estimate every term in the right-hand side. This will lead us to the desired result.

A series of computations imply that

$$\begin{aligned}
|w\beta\nabla\Psi| &\leq \Psi(1-f)w^2 + \frac{c_1 f^4}{\rho^4(1-f)^3}\,,\\
\frac{|\nabla\Psi|^2}{\Psi}\,w &\leq \frac{1}{8}\,\Psi w^2 + \frac{c_1}{\rho^4}\,,\\
-(\Delta\Psi)w &\leq \frac{1}{8}\Psi w^2 + \frac{c_1}{\rho^4} + c_1 k^2
\end{aligned}$$

at (x_1, t_1) for some constant $c_1 > 0$; see [33, 35]. Here, we have used the inequality for the weighted arithmetic mean and the weighted geometric mean, as well as the properties of the function $\bar{\Psi}(r,t)$ given by Lemma 11. Our next mission is to find a suitable bound for $(\Psi_t w)(x_1, t_1)$.

It is clear that

$$\begin{aligned}
(\Psi_t w)(x_1, t_1) = {} & \frac{\partial\bar{\Psi}}{\partial t}(\operatorname{dist}(x_1, x_0, t_1), t_1) w(x_1, t_1)\\
& + \frac{\partial\bar{\Psi}}{\partial r}(\operatorname{dist}(x_1, x_0, t_1), t_1)\left(\frac{\partial}{\partial t}\operatorname{dist}(x_1, x_0, t_1)\right) w(x_1, t_1).
\end{aligned} \tag{2.3.4}$$

We also observe that

$$\left|\frac{\partial\bar{\Psi}}{\partial t}(\operatorname{dist}(x_1, x_0, t_1), t_1)\right| w(x_1, t_1) \leq \frac{1}{16}\left(\Psi w^2\right)(x_1, t_1) + \frac{c_2}{\tau^2}$$

for a positive constant c_2. Because the function $\bar{\Psi}(r,t)$ satisfies the conditions listed in Lemma 11, the inequality

$$\left|\frac{\partial\bar{\Psi}}{\partial r}(\operatorname{dist}(x_1, x_0, t_1), t_1)\right| \leq \frac{C_{\frac{1}{2}}}{\rho}\,\Psi^{\frac{1}{2}}(x_1, t_1) \tag{2.3.5}$$

holds with $C_{\frac{1}{2}} > 0$. It remains to estimate the derivative of the distance. Utilizing the assumptions of the theorem, we conclude that

$$\left|\frac{\partial}{\partial t}\operatorname{dist}(x_1, x_0, t_1)\right| \leq \sup \int_0^{\operatorname{dist}(x_1,x_0,t_1)} \left|\operatorname{Ric}\left(\frac{d}{ds}\zeta(s), \frac{d}{ds}\zeta(s)\right)\right| ds \leq k \operatorname{dist}(x_1, x_0, t_1) \leq k\rho. \tag{2.3.6}$$

In this particular formula, Ric designates the Ricci curvature of $g(x, t_1)$. The supremum is taken over all the minimal geodesics $\zeta(s)$, with respect to $g(x, t_1)$, that connect x_0 to x_1 and are parametrized by arclength; see, e.g., [15, Proof of Lemma 8.28]. It now becomes clear that

$$\Psi_t w \leq \frac{1}{16}\Psi w^2 + \frac{c_2}{\tau^2} + C_{\frac{1}{2}} k w \Psi^{\frac{1}{2}} \leq \frac{1}{8}\Psi w^2 + \frac{c_2}{\tau^2} + c_3 k^2$$

at (x_1, t_1) for some $c_3 > 0$. We have thus found estimates for every term in the right-hand side of (2.3.3). We will combine them all, and the assertion of the theorem will shortly follow.

Given the preceding considerations, formula (2.3.3) implies

$$\Psi(1-f)w^2 \leq \frac{c_4 f^4}{\rho^4(1-f)^3} + \frac{1}{2}\Psi w^2 + \frac{c_4}{\rho^4} + \frac{c_4}{\tau^2} + c_4 k^2$$

at the point (x_1, t_1). The constant c_4 here equals $\max\{3c_1, c_2, c_1 + c_3\}$. Since $f(x,t) \leq 0$ and $\frac{f^4}{(1-f)^4} \leq 1$, we can conclude that

$$\Psi w^2 \leq \frac{c_4 f^4}{\rho^4(1-f)^4} + \frac{1}{2}\Psi w^2 + \frac{c_4}{\rho^4} + \frac{c_4}{\tau^2} + c_4 k^2,$$
$$\Psi^2 w^2 \leq \Psi w^2 \leq \frac{4c_4}{\rho^4} + \frac{2c_4}{\tau^2} + 2c_4 k^2$$

at (x_1, t_1). Because $\Psi(x, \tau) = 1$ when $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$, the estimate

$$w(x,\tau) = (\Psi w)(x,\tau) \leq (\Psi w)(x_1, t_1) \leq \frac{C^2}{\rho^2} + \frac{C^2}{\tau} + C^2 k$$

holds with $C = \sqrt{2\sqrt{c_4}}$ for all $x \in M$ such that $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$. Recalling the definition of $w(x,t)$ and the fact that $\tau \in (0, T]$ was chosen arbitrarily, we obtain the inequality

$$\frac{|\nabla f(x,t)|}{1 - f(x,t)} \leq C\left(\frac{1}{\rho} + \frac{1}{\sqrt{t}} + \sqrt{k}\right)$$

for $(x,t) \in B_{\frac{\rho}{2},T}$ provided $t \neq 0$. The assertion of the theorem follows by an elementary computation.

Our next step is to assume M is compact and state a global gradient estimate for the function $u(x,t)$. This result was previously established in [35, 10].

Theorem 14 (Q. Zhang [35], X. Cao and R. Hamilton [10]) *Suppose the manifold M is compact, and let $(M, g(x,t))_{t\in[0,T]}$ be a solution to the Ricci flow* (2.2.1)*. Assume a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfies the heat equation* (2.2.2)*. Then the estimate*

$$\frac{|\nabla u|}{u} \leq \sqrt{\frac{1}{t} \log \frac{A}{u}}, \qquad x \in M,\ t \in (0,T], \tag{2.3.7}$$

holds with $A = \sup_M u(x,0)$.

Remark The maximum principle implies that A is actually equal to $\sup_{M\times[0,T]} u(x,t)$. This explains why the right-hand side of (2.3.7) is well-defined.

Proof. Consider the function $P = t\frac{|\nabla u|^2}{u} - u \log \frac{A}{u}$ on the set $M \times [0,T]$. It is clear that $P(x,0)$ is nonpositive for every $x \in M$. A computation shows that

$$\left(\Delta - \frac{\partial}{\partial t}\right) P = t\left(\Delta - \frac{\partial}{\partial t}\right)\left(\frac{|\nabla u|^2}{u}\right) = 2\frac{t}{u}\sum_{i,j=1}^{n}\left(u_{ij} - \frac{u_i u_j}{u}\right)^2 \geq 0, \qquad x \in M,\ t \in [0,T].$$

In accordance with the maximum principle, this implies $P(x,t)$ is nonpositive for all $(x,t) \in M \times [0,T]$. The desired assertion follows immediately.

2.4 Space-time gradient estimates

This section establishes Li-Yau-type inequalities for system (2.2.1)–(2.2.2). We will obtain a local and a global estimate. The following lemma will be important to our considerations; cf. Lemma 1 in [32, Chapter IV].

Lemma 15 *Suppose* $(M, g(x,t))_{t\in[0,T]}$ *is a complete solution to the Ricci flow* (2.2.1). *Assume that* $-k_1 g(x,t) \le \mathrm{Ric}(x,t) \le k_2 g(x,t)$ *for some* $k_1, k_2 > 0$ *and all* $(x,t) \in B_{\rho,T}$. *Suppose* $u : M \times [0,T] \to \mathbb{R}$ *is a smooth positive function satisfying the heat equation* (2.2.2). *Given* $\alpha \ge 1$, *define* $f = \log u$ *and* $F = t\left(|\nabla f|^2 - \alpha f_t\right)$. *The estimate*

$$\left(\Delta - \frac{\partial}{\partial t}\right) F \ge -2\nabla f \nabla F + \frac{2a\alpha t}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right) - 2k_1 \alpha t |\nabla f|^2 - \frac{\alpha t n}{2b}\max\left\{k_1^2, k_2^2\right\}, \qquad (x,t) \in B_{\rho,T}, \tag{2.4.1}$$

holds for any $a, b > 0$ *such that* $a + b = \frac{1}{\alpha}$.

Proof. We begin by finding a convenient bound on ΔF. Observe that

$$\Delta F = t\left(2\sum_{i,j=1}^{n}\left(f_{ij}^2 + 2 f_j f_{jii}\right) - \alpha \Delta(f_t)\right), \qquad x \in M,\ t \in [0,T].$$

Our assumption on the Ricci curvature of M implies the inequality

$$\begin{aligned}\sum_{i,j=1}^{n} f_j f_{jii} &= \sum_{i,j=1}^{n}\left(f_j f_{iij} + R_{ij} f_i f_j\right)\\ &= \nabla f \nabla(\Delta f) + \mathrm{Ric}(\nabla f, \nabla f) \ge \nabla f \nabla(\Delta f) - k_1 |\nabla f|^2\end{aligned}$$

at an arbitrary point $(x,t) \in B_{\rho,T}$. Using (2.2.1), we can show that

$$\Delta(f_t) = (\Delta f)_t - 2\sum_{i,j=1}^{n} R_{ij} f_{ij}.$$

Consequently, the estimate

$$\Delta F \ge t\left(2\sum_{i,j=1}^{n}\left(f_{ij}^2 + 2\alpha R_{ij} f_{ij}\right) + 2\nabla f \nabla(\Delta f) - 2k_1 |\nabla f|^2 - \alpha(\Delta f)_t\right)$$

holds at $(x,t) \in B_{\rho,T}$. Our next step is to find a suitable bound on those terms in the right-hand side that involve f_{ij}. We do so by completing the square. More specifically, observe that

$$\sum_{i,j=1}^{n}\left(f_{ij}^2+\alpha R_{ij}f_{ij}\right)=\sum_{i,j=1}^{n}\left((a\alpha+b\alpha)f_{ij}^2+\alpha R_{ij}f_{ij}\right)$$
$$=\sum_{i,j=1}^{n}\left(a\alpha f_{ij}^2+\alpha\left(\sqrt{b}\,f_{ij}+\frac{R_{ij}}{2\sqrt{b}}\right)^2-\frac{\alpha}{4b}R_{ij}^2\right)\geq\sum_{i,j=1}^{n}\left(a\alpha f_{ij}^2-\frac{\alpha}{4b}R_{ij}^2\right)$$

at $(x,t)\in B_{\rho,T}$ for any $a,b>0$ such that $a+b=\frac{1}{\alpha}$. Employing the standard inequality

$$\sum_{i,j=1}^{n}f_{ij}^2\geq\frac{(\Delta f)^2}{n}$$

and the assumptions of the lemma, we obtain the estimate

$$\sum_{i,j=1}^{n}\left(f_{ij}^2+\alpha R_{ij}f_{ij}\right)\geq\frac{a\alpha}{n}(\Delta f)^2-\frac{\alpha n}{4b}\max\left\{k_1^2,k_2^2\right\},\qquad(x,t)\in B_{\rho,T}.$$

It is easy to conclude that

$$\Delta F\geq t\left(\frac{2a\alpha}{n}(\Delta f)^2+2\nabla f\nabla(\Delta f)-2k_1|\nabla f|^2-\alpha(\Delta f)_t-\frac{\alpha n}{2b}\max\left\{k_1^2,k_2^2\right\}\right)$$
$$=\frac{2a\alpha t}{n}\left(f_t-|\nabla f|^2\right)^2+2t\nabla f\nabla\left(f_t-|\nabla f|^2\right)$$
$$-2k_1t|\nabla f|^2-\alpha t\left(f_t-|\nabla f|^2\right)_t-\frac{\alpha tn}{2b}\max\left\{k_1^2,k_2^2\right\}\tag{2.4.2}$$

in the set $B_{\rho,T}$.

Formula (2.4.2) provides us with a convenient bound on ΔF. Let us now include the derivative of F in $t\in[0,T]$ into our considerations. One easily computes

$$\frac{\partial F}{\partial t}=|\nabla f|^2-\alpha f_t+t\left(|\nabla f|^2-\alpha f_t\right)_t.$$

Subtracting this from (2.4.2), we see that the inequality

$$\left(\Delta-\frac{\partial}{\partial t}\right)F\geq\frac{2a\alpha t}{n}\left(f_t-|\nabla f|^2\right)^2+2t\nabla f\nabla\left(f_t-|\nabla f|^2\right)-2k_1t|\nabla f|^2$$
$$-\frac{\alpha tn}{2b}\max\left\{k_1^2,k_2^2\right\}-\left(|\nabla f|^2-\alpha f_t\right)+(\alpha-1)t\left(|\nabla f|^2\right)_t$$

holds in the set $B_{\rho,T}$. In order to arrive to (2.4.1) from here, we need to estimate $\left(|\nabla f|^2\right)_t$. The Ricci flow equation (2.2.1) and the assumptions of the lemma imply

$$\left(|\nabla f|^2\right)_t = 2\nabla f \nabla(f_t) + 2\operatorname{Ric}(\nabla f, \nabla f) \geq 2\nabla f \nabla(f_t) - 2k_1|\nabla f|^2$$

at $(x, t) \in B_{\rho,T}$. As a consequence,

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) F \geq{} & \frac{2a\alpha t}{n}\left(f_t - |\nabla f|^2\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right) \\ & - \frac{\alpha t n}{2b}\max\left\{k_1^2, k_2^2\right\} - 2t\nabla f \nabla\left(|\nabla f|^2 - \alpha f_t\right) - 2k_1\alpha t|\nabla f|^2\end{aligned}$$

in $B_{\rho,T}$. The desired assertion follows immediately.

With Lemma 15 at hand, we are ready to establish the local space-time gradient estimate. We will also make use of arguments from the proof of Theorem 4.2 in [32, Chapter IV]. Recall that n designates the dimension of M.

Theorem 16 *Let $(M, g(x,t))_{t\in[0,T]}$ be a complete solution to the Ricci flow* (2.2.1). *Suppose $-k_1 g(x,t) \leq \operatorname{Ric}(x,t) \leq k_2 g(x,t)$ for some $k_1, k_2 > 0$ and all $(x,t) \in B_{\rho,T}$. Consider a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ solving the heat equation* (2.2.2). *There exists a constant C' that depends only on the dimension of M and satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq C'\alpha^2\left(\frac{\alpha^2}{\rho^2(\alpha-1)} + \frac{1}{t} + \max\{k_1, k_2\}\right) + \frac{nk_1\alpha^3}{\alpha - 1} \tag{2.4.3}$$

for all $\alpha > 1$ and all $(x,t) \in B_{\frac{\rho}{2},T}$ with $t \neq 0$.

Proof. We preserve the notation $f = \log u$ and $F = t\left(|\nabla f|^2 - \alpha f_t\right)$ from Lemma 15. Our strategy in this proof will be similar to that in the proof of Theorem 12. The role of the function $w(x,t)$ now goes to the function $F(x,t)$.

Let us pick $\tau \in (0, T]$ and fix $\bar{\Psi}(r,t)$ satisfying the conditions of Lemma 11. Define $\Psi : M \times [0,T] \to \mathbb{R}$ by setting

$$\Psi(x,t) = \bar{\Psi}(\operatorname{dist}(x, x_0, t), t).$$

We will establish (2.4.3) at (x, τ) for $x \in M$ such that $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$. This will complete the proof. Our plan is to estimate $\left(\frac{\partial}{\partial t} - \Delta\right)(\Psi F)$ and analyze the result at a point where the function ΨF attains its maximum. The required conclusion will follow therefrom.

Lemma 15 and some straightforward computations imply

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi F) \geq& -2\nabla f \nabla(\Psi F) + 2F\nabla f \nabla \Psi \\ &+ \left(\frac{2a\alpha t}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right) - 2k_1 \alpha t |\nabla f|^2 - \frac{\alpha t n}{2b}\bar{k}^2\right)\Psi \\ &+ (\Delta\Psi)F + 2\frac{\nabla\Psi}{\Psi}\nabla(\Psi F) - 2\frac{|\nabla\Psi|^2}{\Psi}F - \frac{\partial \Psi}{\partial t}F \qquad (2.4.4)\end{aligned}$$

with $\bar{k} = \max\{k_1, k_2\}$. This inequality holds in the part of $B_{\rho,T}$ where $\Psi(x, t)$ is smooth and strictly positive. Let (x_1, t_1) be a maximum point for the function ΨF in the set $\{(x, t) \in M \times [0, \tau] \,|\, \mathrm{dist}(x, x_0, t) \leq \rho\}$. We may assume $(\Psi F)(x_1, t_1) > 0$ without loss of generality. Indeed, if this is not the case, then $F(x, \tau) \leq 0$ and (2.4.3) is evident at (x, τ) whenever $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$. We may also assume that $\Psi(x, t)$ is smooth at (x_1, t_1) due to a standard trick explained, for example, in [32, page 21]. Since (x_1, t_1) is a maximum point, the equalities $\Delta(\Psi F)(x_1, t_1) \leq 0$, $\nabla(\Psi F)(x_1, t_1) = 0$, and $(\Psi F)_t(x_1, t_1) \geq 0$ hold true. Combined with (2.4.4), they yield

$$\begin{aligned}0 \geq& \ 2F\nabla f \nabla\Psi + \left(\frac{2a\alpha t_1}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right) - 2k_1\alpha t_1|\nabla f|^2 - \frac{\alpha t_1 n}{2b}\bar{k}^2\right)\Psi \\ &+ (\Delta\Psi)F - 2\frac{|\nabla\Psi|^2}{\Psi}F - \frac{\partial\Psi}{\partial t}F \qquad (2.4.5)\end{aligned}$$

at (x_1, t_1). We will now use (2.4.5) to show that a certain quadratic expression in ΨF is nonpositive. The desired result will then follow.

Let us recall Lemma 11 and apply the Laplacian comparison theorem to conclude that

$$-\frac{|\nabla\Psi|^2}{\Psi} \geq -\frac{C_{\frac{1}{2}}^2}{\rho^2},$$
$$\Delta\Psi \geq -\frac{C_{\frac{1}{2}}}{\rho^2} - \frac{C_{\frac{1}{2}}\Psi^{\frac{1}{2}}}{\rho}(n-1)\sqrt{k_1}\,\coth\left(\sqrt{k_1}\,\rho\right) \geq -\frac{d_1}{\rho^2} - \frac{d_1\Psi^{\frac{1}{2}}}{\rho}\sqrt{k_1}$$

at the point (x_1, t_1) with d_1 a positive constant depending on n. There exists $\bar{C} > 0$ such that the inequality

$$-\frac{\partial\Psi}{\partial t} \geq -\frac{\bar{C}\Psi^{\frac{1}{2}}}{\tau} - C_{\frac{1}{2}}\bar{k}\Psi^{\frac{1}{2}}$$

holds true; cf. (2.3.4), (2.3.5), and (2.3.6). Using these observations along with (2.4.5), we find the estimate

$$0 \geq -2F|\nabla f||\nabla\Psi| + \left(\frac{2a\alpha t_1}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right) - 2k_1\alpha t_1|\nabla f|^2 - \frac{\alpha t_1 n}{2b}\,\bar{k}^2\right)\Psi$$
$$+ d_2\left(-\frac{1}{\rho^2} - \frac{\Psi^{\frac{1}{2}}}{\rho}\sqrt{k_1} - \frac{\Psi^{\frac{1}{2}}}{\tau} - \bar{k}\Psi^{\frac{1}{2}}\right)F$$

at (x_1, t_1). Here, d_2 is equal to $\max\left\{3d_1, C_{\frac{1}{2}}, 3C_{\frac{1}{2}}^2, \bar{C}\right\}$. If one further multiplies by $t\Psi$ and makes a few elementary manipulations, one will obtain

$$0 \geq -2t_1F\frac{C_{\frac{1}{2}}\Psi^{\frac{3}{2}}}{\rho}|\nabla f| + \frac{2t_1^2}{n}\left(a\alpha\left(\Psi|\nabla f|^2 - \Psi f_t\right)^2 - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\,\bar{k}^2\Psi^2\right)$$
$$+ d_2t_1\left(-\frac{1}{\rho^2} - \frac{\sqrt{k_1}}{\rho} - \frac{1}{\tau} - \bar{k}\right)(\Psi F) - \Psi F \tag{2.4.6}$$

at (x_1, t_1). Our next step is to estimate the first two terms in the right-hand side. In order to do so, we need a few auxiliary pieces of notation.

Define $y = \Psi|\nabla f|^2$ and $z = \Psi f_t$. It is clear that $y^{\frac{1}{2}}(y - \alpha z) = \frac{\Psi^{\frac{3}{2}}F|\nabla f|}{t}$ when $t \neq 0$, which yields

$$-2tF\frac{C_{\frac{1}{2}}\Psi^{\frac{3}{2}}}{\rho}|\nabla f| + \frac{2t^2}{n}\left(a\alpha\left(\Psi|\nabla f|^2 - \Psi f_t\right)^2 - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\,\bar{k}^2\Psi^2\right)$$
$$\geq \frac{2t^2}{n}\left(a\alpha(y-z)^2 - nk_1\alpha y - \frac{n^2\alpha}{4b}\,\bar{k}^2\Psi^2 - \frac{nC_{\frac{1}{2}}}{\rho}\,y^{\frac{1}{2}}(y - \alpha z)\right).$$

Let us observe that

$$(y-z)^2 = \frac{1}{\alpha^2}(y-\alpha z)^2 + \frac{(\alpha-1)^2}{\alpha^2}y^2 + \frac{2(\alpha-1)}{\alpha^2}y(y-\alpha z)$$

and plug this into the previous estimate. Regrouping the terms and applying the inequality $\kappa_1 v^2 - \kappa_2 v \geq -\frac{\kappa_2^2}{4\kappa_1}$ valid for $\kappa_1, \kappa_2 > 0$ and $v \in \mathbb{R}$, we obtain

$$\begin{aligned}
-2tF\frac{C_{\frac{1}{2}}\Psi^{\frac{3}{2}}}{\rho}|\nabla f| + \frac{2t^2}{n}\left(a\alpha\left(\Psi|\nabla f|^2 - \Psi f_t\right)^2 - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\bar{k}^2\Psi^2\right)& \\
\geq \frac{2t^2}{n}\left(\frac{a}{\alpha}(y-\alpha z)^2 - \frac{n^2k_1^2\alpha^3}{4a(\alpha-1)^2} - \frac{n^2 d_2\alpha}{8a\rho^2(\alpha-1)}(y-\alpha z) - \frac{n^2\alpha}{4b}\bar{k}^2\Psi^2\right)&.
\end{aligned}$$

Because $t(y-\alpha z) = \Psi F$ by definition, (2.4.6) now implies

$$\begin{aligned}
0 &\geq \frac{2a}{n\alpha}(\Psi F)^2 + \left(-\frac{nd_2t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + 1 + \rho\sqrt{\bar{k}} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) - 1\right)(\Psi F) \\
&\quad - \frac{nk_1^2\alpha^3}{2a(\alpha-1)^2}t_1^2 - \frac{\alpha n}{2b}t_1^2\bar{k}^2\Psi^2 \\
&\geq \frac{2a}{n\alpha}(\Psi F)^2 + \left(-\frac{d_3t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) - 1\right)(\Psi F) \\
&\quad - \frac{nk_1^2\alpha^3}{2a(\alpha-1)^2}t_1^2 - \frac{\alpha n}{2b}t_1^2\bar{k}^2\Psi^2
\end{aligned}$$

at (x_1, t_1) with $d_3 = 4nd_2$. The expression in the last two lines is a polynomial in ΨF of degree 2. Consequently, in accordance with the quadratic formula,

$$\Psi F \leq \frac{n\alpha}{2a}\left(\frac{d_3t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) + 1 + \frac{k_1\alpha}{\alpha-1}t_1 + \sqrt{\frac{a}{b}}\,t_1\bar{k}\Psi\right)$$

at (x_1, t_1). We will now use this conclusion to obtain a bound on $F(x,\tau)$ for an appropriate range of $x \in M$.

Recall that $\Psi(x,\tau) = 1$ whenever $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$. Besides, (x_1, t_1) is a maximum point for ΨF in the set $\{(x,t) \in M \times [0,\tau] \,|\, \mathrm{dist}(x, x_0, t) \leq \rho\}$. Hence

$$\begin{aligned}
F(x,\tau) &= (\Psi F)(x,\tau) \leq (\Psi F)(x_1, t_1) \\
&\leq \frac{n\alpha d_3\tau}{2a\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) + \frac{n\alpha}{2a} + \frac{nk_1\alpha^2}{2a(\alpha-1)}\tau + \frac{\alpha\tau n\bar{k}}{2}\sqrt{\frac{1}{ab}}
\end{aligned}$$

for all $x \in M$ such that $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$. Since $\tau \in (0, T]$ was chosen arbitrarily, this formula implies

$$\left(|\nabla f|^2 - \alpha f_t\right)(x,t) \le \frac{\alpha d_4}{a\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{t} + \rho^2\bar{k}\right) + \frac{nk_1\alpha^2}{2a(\alpha-1)} + \frac{\alpha n\bar{k}}{2}\sqrt{\frac{1}{ab}}\,, \qquad (x,t) \in B_{\frac{\rho}{2},T},$$

with $d_4 = \max\{nd_3, n\}$ as long as $t \neq 0$. If we set $a = \frac{1}{2\alpha}$, note that $b = \frac{1}{\alpha} - a$, and define the constant C' appropriately, estimate (2.4.3) will follow by a straightforward computation.

Remark The value $\frac{1}{2\alpha}$ for the parameter a in the proof of the theorem might not be optimal. It is possible that a different a will lead to a sharper estimate.

Let us now consider the case where the manifold M is compact. We will present a global estimate on $u(x,t)$ demanding that the Ricci curvature of M be nonnegative. A related inequality for (2.2.1)–(2.2.2) may be found in [16].

Theorem 17 *Suppose the manifold M is compact and $(M, g(x,t))_{t\in[0,T]}$ is a solution to the Ricci flow* (2.2.1). *Assume that $0 \le \mathrm{Ric}(x,t) \le kg(x,t)$ for some $k > 0$ and all $(x,t) \in M \times [0,T]$. Consider a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfying the heat equation* (2.2.2). *The estimate*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le kn + \frac{n}{2t} \tag{2.4.7}$$

holds for all $(x,t) \in M \times (0,T]$.

Proof. As before, we write f instead of $\log u$. It will be convenient for us to denote $F_1 = t\left(|\nabla f|^2 - f_t\right)$. Fix $\tau \in (0,T]$ and choose a point $(x_0, t_0) \in M \times [0,\tau]$ where F_1 attains its maximum on $M \times [0,\tau]$. Our first step is to show that

$$F_1(x_0,t_0) \le t_0 kn + \frac{n}{2}\,. \tag{2.4.8}$$

The assertion of the theorem will follow therefrom.

If $t_0 = 0$, then $F_1(x, t_0)$ is equal to 0 for every $x \in M$ and estimate (2.4.8) becomes evident. Consequently, we can assume $t_0 > 0$ without loss of generality. Lemma 15 and our conditions on the Ricci curvature of M imply the inequality

$$\left(\Delta - \frac{\partial}{\partial t}\right) F_1 \geq -2\nabla f \nabla F_1 + \frac{2a}{n}\frac{F_1^2}{t_0} - \frac{F_1}{t_0} - \frac{t_0 n}{2(1-a)} k^2$$

for all $a \in (0, 1)$ at the point (x_0, t_0). Now recall that F_1 attains its maximum at (x_0, t_0). This tells us that $\Delta F_1(x_0, t_0) \leq 0$, $\frac{\partial}{\partial t} F_1(x_0, t_0) \geq 0$, and $\nabla F_1(x_0, t_0) = 0$. In consequence, the estimate

$$\frac{2a}{n}\frac{F_1^2}{t_0} - \frac{F_1}{t_0} - \frac{t_0 n}{2(1-a)} k^2 \leq 0$$

holds at (x_0, t_0), and the quadratic formula yields

$$F_1(x_0, t_0) \leq \frac{n}{4a}\left(1 + \sqrt{1 + \frac{4at_0^2}{1-a} k^2}\right).$$

The expression in the right-hand side is minimized in $a \in (0, 1)$ when a is equal to $\frac{1+kt_0}{1+2kt_0}$. Plugging this value of a into the above inequality, we arrive at (2.4.8).

Only a simple argument is now needed to complete the proof. The fact that (x_0, t_0) is a maximum point for F_1 on $M \times [0, \tau]$ enables us to conclude that

$$F_1(x, \tau) \leq F_1(x_0, t_0) \leq t_0 kn + \frac{n}{2} \leq \tau kn + \frac{n}{2}$$

for all $x \in M$. Therefore, the estimate

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq kn + \frac{n}{2\tau}$$

holds at (x, τ). Because the number $\tau \in (0, T]$ can be chosen arbitrarily, the assertion of the theorem follows.

2.5 Harnack inequalities

The last part of this chapter will consist of two Harnack inequalities for (2.2.1)–(2.2.2). These may be viewed as applications of Theorems 16 and 17; cf., for example, [32, Chapter IV]. One can find other Harnack inequalities for (2.2.1)–(2.2.2) in the papers [16, 28]. We first introduce a piece of notation. Given $x_1, x_2 \in M$ and $t_1, t_2 \in (0, T)$ satisfying $t_1 < t_2$, define

$$\Gamma(x_1, t_1, x_2, t_2) = \inf \int_{t_1}^{t_2} \left| \frac{d}{dt}\gamma(t) \right|^2 dt.$$

The infimum is taken over the set $\Theta(x_1, t_1, x_2, t_2)$ of all the smooth paths $\gamma : [t_1, t_2] \to M$ that connect x_1 to x_2. We remind the reader that the norm $|\cdot|$ depends on t. Let us now present a lemma. It will be the key to proving of our results.

Lemma 18 *Suppose* $(M, g(x,t))_{t\in[0,T]}$ *is a complete solution to the Ricci flow* (2.2.1). *Let* $u : M \times [0, T] \to \mathbb{R}$ *be a smooth positive function satisfying the heat equation* (2.2.2). *Define* $f = \log u$ *and assume that*

$$\frac{\partial f}{\partial t} \geq \frac{1}{A_1}\left(|\nabla f|^2 - A_2 - \frac{A_3}{t} \right), \qquad x \in M,\ t \in (0, T],$$

for some $A_1, A_2, A_3 > 0$. *Then the inequality*

$$u(x_2, t_2) \geq u(x_1, t_1) \left(\frac{t_2}{t_1} \right)^{-\frac{A_3}{A_1}} \exp\left(-\frac{A_1}{4}\Gamma(x_1, t_1, x_2, t_2) - \frac{A_2}{A_1}(t_2 - t_1) \right)$$

holds for all $(x_1, t_1) \in M \times (0, T)$ *and* $(x_2, t_2) \in M \times (0, T)$ *such that* $t_1 < t_2$.

PROOF. The method we use is rather traditional; see, for example, [32, Chapter IV] and [10]. Consider a path $\gamma(t) \in \Theta(x_1, t_1, x_2, t_2)$. We begin by computing

$$\begin{aligned}
\frac{d}{dt}f(\gamma(t),t) &= \nabla f(\gamma(t),t)\frac{d}{dt}\gamma(t) + \frac{\partial}{\partial s}f(\gamma(t),s)|_{s=t} \\
&\geq -|\nabla f(\gamma(t),t)|\left|\frac{d}{dt}\gamma(t)\right| + \frac{1}{A_1}\left(|\nabla f(\gamma(t),t)|^2 - A_2 - \frac{A_3}{t}\right) \\
&\geq -\frac{A_1}{4}\left|\frac{d}{dt}\gamma(t)\right|^2 - \frac{1}{A_1}\left(A_2 + \frac{A_3}{t}\right), \qquad t \in [t_1, t_2].
\end{aligned}$$

The last step is a consequence of the inequality $\kappa_1 v^2 - \kappa_2 v \geq -\frac{\kappa_2^2}{4\kappa_1}$ valid for $\kappa_1, \kappa_2 > 0$ and $v \in \mathbb{R}$. The above implies

$$\begin{aligned}
f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{d}{dt}f(\gamma(t),t)\,dt \\
&\geq -\frac{A_1}{4}\int_{t_1}^{t_2}\left|\frac{d}{dt}\gamma(t)\right|^2\,dt - \frac{A_2}{A_1}(t_2-t_1) - \frac{A_3}{A_1}\ln\frac{t_2}{t_1}.
\end{aligned}$$

The assertion of the lemma follows by exponentiating.

We are ready to formulate our Harnack inequalities for (2.2.1)–(2.2.2). The first one applies on noncompact manifolds. The second one does not, but it provides a more explicit estimate.

Theorem 19 *Let* $(M, g(x,t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (2.2.1). *Assume that* $-k_1 g(x,t) \leq \mathrm{Ric}(x,t) \leq k_2 g(x,t)$ *for some* $k_1, k_2 > 0$ *and all* $(x,t) \in M \times [0,T]$. *Suppose a smooth positive function* $u : M \times [0,T] \to \mathbb{R}$ *satisfies the heat equation* (2.2.2). *Given* $\alpha > 1$, *the estimate*

$$u(x_2,t_2) \geq u(x_1,t_1)\left(\frac{t_2}{t_1}\right)^{-C'\alpha} \exp\left(-\frac{\alpha}{4}\Gamma(x_1,t_1,x_2,t_2) - \left(C'\alpha\max\{k_1,k_2\} + \frac{nk_1\alpha^2}{\alpha-1}\right)(t_2-t_1)\right)$$

holds for all $(x_1,t_1) \in M \times (0,T)$ *and* $(x_2,t_2) \in M \times (0,T)$ *such that* $t_1 < t_2$. *The constant* C' *comes from Theorem 16.*

Proof. Letting ρ go to infinity in (2.4.3), we conclude that

$$\frac{u_t}{u} \geq \frac{1}{\alpha}\left(\frac{|\nabla u|^2}{u^2} - \frac{C'\alpha^2}{t} - \left(C'\alpha^2\max\{k_1,k_2\} + \frac{nk_1\alpha^3}{\alpha-1}\right)\right)$$

on $M \times (0, T]$. The desired assertion is now a consequence of Lemma 18.

Theorem 20 *Suppose M is compact and $(M, g(x,t))_{t\in[0,T]}$ is a solution to the Ricci flow* (2.2.1). *Assume that $0 \le \mathrm{Ric}(x,t) \le kg(x,t)$ for some $k > 0$ and all $(x,t) \in M \times [0,T]$. Consider a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfying the heat equation* (2.2.2). *The estimate*

$$u(x_2, t_2) \ge u(x_1, t_1) \left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} \exp\left(-\frac{1}{4}\Gamma(x_1, t_1, x_2, t_2) - kn(t_2 - t_1)\right)$$

holds for all $(x_1, t_1) \in M \times (0, T)$ and $(x_2, t_2) \in M \times (0, T)$ as long as $t_1 < t_2$.

PROOF. Theorem 17 implies

$$\frac{u_t}{u} \ge \frac{|\nabla u|^2}{u^2} - kn - \frac{n}{2t}, \qquad x \in M,\ t \in (0, T].$$

One may now use Lemma 18 to complete the proof.

CHAPTER 3

BOUNDS ON THE HEAT KERNEL

3.1 Introduction

Determining bounds for the heat operator on manifolds has been a topic of interest, as it had proven to have many applications. D. Aronson made use of a parabolic Harnack inequality to bound the fundamental solution for a general second-order parabolic operator [2]. Later, in their celebrated paper [25], P. Li and S.-T. Yau derived gradient estimates for positive solutions to the heat equation on closed manifolds with bounded Ricci curvature, from which they obtained Harnack inequalities. Further these inequalities were used to get upper and lower bounds on the heat kernel. They considered manifolds with boundaries, satisfying Dirichlet and Neumann boundary conditions, the heat kernel bounds extending to the boundary when the latter was convex. Later, J. Wang derived in [34] a global version of gradient estimates when the boundary is nonconvex, and he obtained both upper and lower bounds for the heat kernel satisfying Neumann conditions.

In geometric analysis, heat kernel estimates, together with Sobolev imbedding theorems, have been proven useful in the study of Ricci flows, especially in the case with surgeries. Since Sobolev imbeddings and inequalities relate the integrability (in some L^p sense) of the derivative of a function to the integrability of the function itself, they become useful when looking at partial differential equations. They also have proven useful in characterizing the space where the function is defined (for a detailed discussion see, for example, [31]).

In [16], C. Guenther studied the fundamental solution of the linear parabolic operator $L(u) = (\Delta - \frac{\partial}{\partial t} - h)u$, on compact n-dimensional manifolds with time dependent metric, where h is a smooth space-time function. She proved the uniqueness, positivity, the adjoint property and the semigroup property of this operator, which thus behaves like the usual heat kernel. As a particular case ($h = 0$), she obtained the existence and properties of the heat kernel under the Ricci flow.

G. Perelman gave a proof in [29] of the pseudolocality theorem, which states that Euclidean looking regions in closed manifolds evolving by Ricci flow remain localized, under some curvature assumptions. In order to prove this, he obtained a differential Li-Yau-Hamilton type inequality for the fundamental solution of the conjugate heat equation $\Delta u + \frac{\partial}{\partial t} - Ru = 0$, where R is the scalar curvature on the manifold. Later S.-Y. Hsu obtained in [23], by a variation of the method introduced by P. Li, S.-T. Yau and J. Wang, a gradient estimate for the solution of the conjugate heat equation on closed manifolds under Ricci flow, and as a consequence, bounds for its fundamental solution.

Q. Zhang also considered in [35] the conjugate heat equation introduced by Perelman, but after a time reversal: $\Delta u - \frac{\partial}{\partial t} - Ru = 0$, and the metric evolving under forward Ricci flow $\frac{\partial}{\partial t} g(x,t) = 2\,\mathrm{Ric}(x,t)$. He considered a complete manifold, with $\mathrm{Ric}(g(t)) \geq k$ and the injectivity radius $i > 0$. He obtained un upper bound on the fundamental solution of this equation using the Nash method, without any gradient estimate, and his result depends on the best constants in the Sobolev imbedding theorem. Let's just mention that our heat kernel equals the fundamental solution in Q. Zhang's paper (since $G(x,t;y,s)$ satisfies the conjugate heat equation in the (y,s) variables) and our result is an improvement, since there are no conditions on the Ricci curvature or the injectivity radius.

Recently, Q. Zhang and X. Cao characterized in [11] the Type I singularity model of the Ricci flow by means of upper and lower bounds of the fundamental solution of the conjugate heat equation.

In this paper, we obtain a bound on the heat kernel, depending on the best constants in a Sobolev imbedding theorem. We state the Sobolev imbedding theorems and the constants that the result will depend on in section (3.2), while the proof is given in section (3.4). We will conclude the paper with the special case, when the scalar curvature is positive at the initial time (and hence, since the Ricci flow preserves the positivity of the scalar curvature, throughout the manifold at all given times).

We should add that the bound we get is not sharp, and that a long term goal is to get estimates similar to the ones of Li-Yau. This, however, has proved to be much more difficult, due to the changing nature of the metric. Hope comes from the fact that similar gradient estimates on the solution of the heat equation have been found by the author, together with X. Cao and A. Pulemotov, using similar methods (see [6]).

3.2 Setup

We consider an n-dimensional manifold without boundary M, which is compact, connected, oriented and smooth.

For $T > 0$, let $(M, g(x,t)), t \in [0,T]$ be a solution to the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \qquad x \in M,\ t \in [0,T] \tag{3.2.1}$$

The interval that we consider $[0,T]$ is a subset of the interval of short-time existence, hence we won't deal with blow-ups.

We say that a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfies the heat equation if the following holds

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0\,.$$

Here, Δ stands for the Laplacian given by $g(x,t)$. We will use ∇ and $|\cdot|$ to denote the gradient and the norm with respect to $g(x,t)$, respectively. We emphasize that Δ, ∇, and $|\cdot|$ all depend on $t \in [0,T]$. XY denotes the scalar product of the vectors X and Y with respect to the metric $g(x,t)$.

We will denote the heat kernel, i.e. the fundamental solution of the heat operator $\left(\Delta - \frac{\partial}{\partial t}\right)$ by $G(x,t;y,s)$. Let's recall that the fundamental solution of an operator L is a smooth function $G(x,t;y,s) : M \times [0,T] \times M \times [0,T] \to \mathbb{R}$, with $s < t$, which satisfies two properties:

(i) $L(G) = 0$ in (x,t) for $(x,t) \neq (y,s)$,

(ii) $\lim_{t \to s} G(.,t;y,s) = \delta_y$ for every y, where δ_y is the Dirac delta function.

Guenther proved the existence and studied the properties of the fundamental solution for the operator $L(u) = \left(\Delta - \frac{\partial}{\partial t} - h(x,t)\right) u$ on a compact manifold whose metric evolves under the Ricci flow ($h(x,t)$ is a smooth function) [16, Theorem 2.1]. In particular, if $h(x,t) = 0$ we get the existence of the heat kernel.

During the computations, we will sometimes drop the arguments $(x, t; y, s)$, as it will be clear with respect to which variables we are considering the measure over which we are integrating.

Our proof relies on two Sobolev imbedding theorems, which are stated below. Since the manifold is compact, there will be no assumption on the injectivity radius or on the Ricci curvature.

In [22], Hebey and Vaugon (improving a result by Aubin [4]) proved the following:

Theorem 21 *Let M^n be a compact Riemannian manifold. If $1 \leq q \leq n$, then for any $q' \in [1, q]$ and for any $r > 1$, there exists a positive constant $B = B(g, n)$ such that for any $\psi \in W^{1,q}(M)$ (the Sobolev space of weakly differentiable functions) the following is true:*

$$||\psi||_p^r \leq K(n, q)^r ||\nabla\psi||_q^r + B||\psi||_{q'}^r .$$

Here $K(n, q)$ is the best constant in the Sobolev imbedding (inequality) in $\mathbb{R}^n$ and $p = (nq)/(n - q)$.

Along the Ricci flow, Zhang proved the following uniform Sobolev inequality in [36]:

Theorem 22 *Let M^n be a compact Riemannian manifold, with $n \geq 3$ and let $(M, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow* (3.2.1). *Let A and B be positive numbers such that for $(M, g(0))$ the following Sobolev inequality holds: for any $v \in W^{1,2}(M, g(0))$,*

$$\left(\int_M |v|^{\frac{2n}{n-2}} \, d\mu_{g(0)}\right)^{\frac{n-2}{n}} \leq A \int_M |\nabla v|^2 \, d\mu_{g(0)} + B \int_M v^2 \, d\mu_{g(0)} .$$

Then there exist positive functions $A(t)$, $B(t)$ depending only on the initial metric $g(0)$, A, B, and t such that, for all $v \in W^{1,2}(M, g(t))$, $t > 0$, the following holds

$$\left(\int_M |v|^{\frac{2n}{n-2}} \, d\mu_{g(t)} \right)^{\frac{n-2}{n}} \leq A(t) \int_M \left(|\nabla v|^2 + \frac{1}{4} R v^2 \right) \, d\mu_{g(t)} + B(t) \int_M v^2 \, d\mu_{g(t)} \, .$$

Here R is the scalar curvature with respect to $g(t)$. Moreover, if $R(x, 0) > 0$, then $A(t)$ and $B(t)$ are independent of t.

3.3 Main result

The main result of this chapter can be stated as follows:

Theorem 23 *Let M^n be a closed Riemannian manifold, with $n \geq 3$ and let $(M, g(x, t)), t \in [0, T]$ be a solution to the Ricci flow* (3.2.1). *Let $G(x, t; y, s)$ be the heat kernel, i.e. fundamental solution for the heat equation* (2.2.2). *Then there exists a positive number C_n, which depends only on the dimension n of the manifold such that:*

$$G(x, t; y, s) \leq \frac{C_n}{\left(\int\limits_s^{\frac{s+t}{2}} \left(\frac{m_0 - c_n \tau}{m_0} \right)^{-2} \frac{e^{\frac{2}{n} H(\tau)}}{A(\tau)} \, d\tau \right)^{\frac{n}{4}} \left(\int\limits_{\frac{s+t}{2}}^{t} \frac{e^{-\frac{2}{n} H(\tau)}}{A(\tau)} \, d\tau \right)^{\frac{n}{4}}} \, ,$$

for $0 \leq s < t \leq T$; here $H(t) = \int\limits_s^t \left[\frac{B(\tau)}{A(\tau)} - \frac{3}{4} \cdot \frac{1}{m_0 - c_n \tau} \right] d\tau$, $1/m_0 = \inf_{t=0} R$ - the infimum of the scalar curvature, taken at time 0, and $A(t)$ and $B(t)$ are positive functions, which depend on the best constant in the Sobolev imbedding theorem.

One can notice that there are no curvature assumptions, just like in [35] and in [23], where the conjugate heat equation was analysed.

The estimate may not seem natural, but in a special case, when the scalar curvature $R(x,0) > 0$ (and thus $R(x,t) > 0$ for any $t \in [0,T]$), one obtains a similar result to the one in the fixed metric case. Let's recall that J. Wang obtained in [34] that the heat kernel on an n-dimensional compact Riemannian manifold M, with fixed metric, is bounded from above by $C(S)(t-s)^{-n/2}$, where $C(S)$ is the Neumann Sobolev constant of M, coming from a Sobolev imbedding theorem. Our corollary exibits a similar bound:

Corollary 24 *Under the same assumptions as in theorem (23), together with the condition that the scalar curvature $R(x,t)$ be (strictly) positive at $t = 0$, there exists a positive number $\tilde{C}_n$, which depends only on the dimension n of the manifold and on the best constant in the Sobolev imbedding theorem in $\mathbb{R}^n$, such that:*

$$G(x,t;y,s) \leq \tilde{C}_n \cdot \frac{1}{(t-s)^{\frac{n}{2}}} \qquad \text{, for } 0 \leq s < t \leq T \, .$$

The exact expression of $\tilde{C}_n$ will be shown in the proof. $\tilde{C}_n$ differs from $C(S)$ (as it appears in [34]).

3.4 Proof

We assume without loss of generality that $s = 0$. By the semigroup property of the heat kernel [16, Theorem 2.6] and the Cauchy-Bunyakovsky-Schwarz inequality we have that:

$$\begin{aligned} G(x,t;y,0) &= \int_M G\left(x,t;z,\frac{t}{2}\right) G\left(z,\frac{t}{2};y,0\right) d\mu_{\left(z,\frac{t}{2}\right)} \\ &\leq \left[\int_M G^2\left(x,t;z,\frac{t}{2}\right) d\mu_{\left(z,\frac{t}{2}\right)}\right]^{1/2} \left[\int_M G^2\left(z,\frac{t}{2};y,0\right) d\mu_{\left(z,\frac{t}{2}\right)}\right]^{1/2} . \end{aligned}$$

The key of the proof consists in determining upper bounds for the following two quantities:

$$\alpha(t) = \int_M G^2(x,t;y,s)d\mu_{(x,t)} \text{ (for } s \text{ fixed)},$$
$$\beta(s) = \int_M G^2(x,t;y,s)d\mu_{(y,s)} \text{ (for } t \text{ fixed)}.$$

Once we have these bounds, the conclusion follows immediately. The strategy to get these bounds consists in finding an ordinary differential inequality for each of the two quantities.

First let's recall that, by definition, G satisfies the heat equation in the (x,t) coordinates

$$\Delta_x G(x,t;y,s) - \partial_t G(x,t;y,s) = 0\,,$$

whereas in the (y,s) it satisfies the conjugate heat equation

$$\Delta_y G(x,t;y,s) + \partial_s G(x,t;y,s) - R(y,s)G(x,t;y,s) = 0\,,$$

here $R(y,s)$ being the scalar curvature, measured with respect to the metric $g(s)$.

We will first deduce a bound on $\alpha(t)$, by finding an inequality involving $\alpha'(t)$ and $\alpha(t)$. Note that we will treat G as being a function of (x,t), the (y,s) part is fixed.

Since $\frac{d}{dt}(d\mu) = -Rd\mu$ (due to the Ricci flow), one has:

$$\begin{aligned}\alpha'(t) = \int_M \left(2\,G\cdot G_t - G^2R\right)\,d\mu_{(x,t)} = 2\int_M G\,(\Delta G)\,d\mu_{(x,t)} - \int_M G^2R\,d\mu_{(x,t)} = \\ -2\int_M |\nabla G|^2\,d\mu_{(x,t)} - \int_M G^2R\,d\mu_{(x,t)} \leq -\int_M [|\nabla G|^2 + RG^2]\,d\mu_{(x,t)}\,. \qquad (3.4.1)\end{aligned}$$

The difficult part will be to estimate $\int_M |\nabla G|^2 d\mu$. The way we proceed is to use the Sobolev imbedding theorem, which gives a relation between $\int_M |\nabla G|^2 d\mu$ and $\int_M G^2 d\mu$, and the Hölder inequality to bound the term involving $G^{2n/(n-2)}$:

$$\int_M G^2 \, d\mu_{(x,t)} \leq \left(\int_M G^{\frac{2n}{n-2}} \, d\mu_{(x,t)} \right)^{\frac{n-2}{n+2}} \left(\int_M G \, d\mu_{(x,t)} \right)^{\frac{4}{n+2}} . \tag{3.4.2}$$

By theorem (21), for $r = 2$, $q = q' = 2$ and $p = 2n/(n-2)$ one gets that at time $t = 0$, the following inequality holds for any $v \in W^{1,2}(M, g(0))$ and for some $B > 0$:

$$\left(\int_M |v|^{\frac{2n}{n-2}} \, d\mu_{g(0)} \right)^{\frac{n-2}{n}} \leq K(n,2)^2 \int_M |\nabla v|^2 \, d\mu_{g(0)} + B \int_M v^2 \, d\mu_{g(0)} ,$$

where $K(n, 2)$ - the best constant in the Sobolev imbedding in $\mathbb{R}$.

From this, by theorem 22 it follows that at any time $t > 0$ within the interval of existence of the solution to the Ricci flow $[0, T]$ one has that for all $v \in W^{1,2}(M, g(t))$:

$$\left(\int_M |v|^{\frac{2n}{n-2}} \, d\mu_{g(t)} \right)^{\frac{n-2}{n}} \leq A(t) \int_M \left(|\nabla v|^2 + \frac{1}{4} R v^2 \right) \, d\mu_{g(t)} + B(t) \int_M v^2 \, d\mu_{g(t)} ,$$

where $A(t)$ is a positive function depending on $g(0)$ and $K(n, 2)^2$, while $B(t)$ is also a positive function, depending on $g(0)$ and B.

Since $G(x, t; ., .) \in W^{1,2}(M, g(t))$ (it is even smooth), then the above holds, so one can relate the RHS of (3.4.2) to the Sobolev inequality:

$$\int_M G^2\, d\mu_{(x,t)} \leq \left(\int_M G^{\frac{2n}{n-2}}\, d\mu_{(x,t)}\right)^{\frac{n-2}{n+2}} \left(\int_M G\, d\mu_{(x,t)}\right)^{\frac{4}{n+2}} \leq$$
$$\leq \left[A(t)\int_M \left(|\nabla G|^2 + \frac{1}{4}RG^2\right) d\mu_{(x,t)} + B(t)\int_M G^2\, d\mu_{(x,t)}\right]^{\frac{n}{n+2}} \left(\int_M G\, d\mu_{(x,t)}\right)^{\frac{4}{n+2}}. \quad (3.4.3)$$

We need to estimate the term $J(t) := \int_M G(x,t;y,s)\, d\mu_{(x,t)}$. By the definition of the fundamental solution, we have that: $\int_M G(x,t;y,s) d\mu_{(y,s)} = 1$, but that's not true if one integrates in (x,t). We will obtain a differential inequality for $J(t)$ and the estimate will follow therefrom.

$$\begin{aligned} J'(t) &= \int_M G_t(x,t;y,s)\, d\mu_{(x,t)} + \int_M G(x,t;y,s)\frac{d}{dt}\, d\mu_{(x,t)} \\ &= \int_M \Delta_x G(x,t;y,s)\, d\mu_{(x,t)} - \int_M G(x,t;y,s)R(x,t)\, d\mu_{(x,t)} \\ &= -\int_M G(x,t;y,s)R(x,t)\, d\mu_{(x,t)}\,, \end{aligned}$$

the first term being 0, since the manifold is compact, without boundary.

The scalar curvature satisfies the following differential inequality (see [15]):

$$\frac{\partial R}{\partial t} - \Delta R - \frac{2}{n}R^2 \geq 0\,.$$

Since the solutions of the ODE $\frac{d\rho}{dt} = \frac{2}{n}\rho^2$ are $\rho(t) = \frac{n}{n\rho(0)^{-1} - 2t}$, by the maximum principle we get a bound on the scalar curvature, for $s \leq \tau \leq t$:

$$R(z,\tau) \geq \frac{n}{n(\inf_{t=0} R)^{-1} - 2\tau} = \frac{1}{(\inf_{t=0} R)^{-1} - \frac{2}{n}\tau} := \frac{1}{m_0 - c_n\tau}$$

(here and later, if $\inf_{t=0} R \geq 0$, then the above is regarded as zero).

Using this lower bound for R (for $\tau \in (s,t]$), we find:

$$J'(\tau) \leq -\frac{1}{m_0 - c_n\tau} J(\tau)\,.$$

After integrating the above from s to t, while noting that by $J(s)$ one understands:

$$J(s) = \lim_{t\to s} \int\limits_M G(x,t;y,s)\, d\mu_{(x,t)} = \int\limits_M \lim_{t\to s} G(x,t;y,s)\, d\mu_{(x,t)} = \int\limits_M \delta_y(x)\, d\mu_{(x,s)} = 1\,,$$

one obtains:

$$J(t) \leq \left(\frac{m_0 - c_n t}{m_0 - c_n s}\right)^{\frac{n}{2}} := (\chi_{t,s})^{\frac{n}{2}}\,.$$

Hence $\int\limits_M G(x,t;y,s)\, d\mu_{(x,t)} \leq (\chi_{t,s}))^{\frac{n}{2}}$ and (3.4.3) becomes:

$$\int\limits_M G^2\, d\mu_{(x,t)} \leq (\chi_{t,s})^{\frac{2n}{n+2}} \left[A(t) \int\limits_M \left(|\nabla G|^2 + \frac{1}{4}RG^2\right)\, d\mu_{(x,t)} + B(t) \int\limits_M G^2\, d\mu_{(x,t)}\right]^{\frac{n}{n+2}}\,.$$

From this it follows immediately that:

$$\int\limits_M |\nabla G|^2\, d\mu_{(x,t)} \geq \frac{1}{\chi_{t,s}^2 A(t)} \left(\int\limits_M G^2\, d\mu_{(x,t)}\right)^{\frac{n+2}{n}} - \frac{B(t)}{A(t)} \int\limits_M G^2\, d\mu_{(x,t)} - \frac{1}{4}\int\limits_M RG^2\, d\mu_{(x,t)}\,.$$

Combining this with the inequality from (3.4.1), one obtains the following differential inequality for $\alpha(t)$:

$$\alpha'(t) \leq -\frac{1}{\chi_{t,s}^2 A(t)} \alpha(t)^{\frac{n+2}{n}} + \frac{B(t)}{A(t)}\alpha(t) - \frac{3}{4}\int\limits_M RG^2 d\mu_{(x,t)}\,.$$

Note that the above is true for any $\tau \in (s,t]$. For the following computation, we will consider t fixed as well. Recall that for $\tau \in (s,t]$, $R(\cdot,\tau) \geq \frac{1}{m_0 - c_n\tau}$. Denoting with:

$$h(\tau) := \frac{B(\tau)}{A(\tau)} - \frac{3}{4} \cdot \frac{1}{m_0 - c_n \tau} \, ,$$

we get:

$$\alpha'(\tau) \leq -\frac{1}{\chi^2_{\tau,s} A(\tau)} \alpha(\tau)^{\frac{n+2}{n}} + h(\tau)\alpha(\tau) \, .$$

Let $H(\tau)$ be an antiderivative of $h(\tau)$. By the integrating factor method, one finds:

$$\left(e^{-H(\tau)}\alpha(\tau)\right)' \leq -\frac{1}{\chi^2(\tau)A(\tau)} \left(e^{-H(\tau)}\alpha(\tau)\right)^{\frac{n+2}{n}} e^{\frac{2}{n}H(\tau)} \, .$$

Since the above is true for any $\tau \in (s, t]$, by integrating from s to t and taking into account that

$$\lim_{\tau \searrow s} \alpha(\tau) = \int_M \lim_{\tau \searrow s} G^2(x, \tau; y, s) \, d\mu(x, \tau) = \int_M \delta_y^2(x) \, d\mu(x, s) = 0 \, ,$$

one obtains the first necessary bound:

$$\alpha(t) \leq \frac{C_n e^{H(t)}}{\left(\int_s^t \frac{e^{\frac{2}{n}H(\tau)}}{\chi^2(\tau)A(\tau)} d\tau\right)^{\frac{n}{2}}} \, ,$$

where $C_n = \left(\frac{2}{n}\right)^{\frac{n}{2}}$.

The next step is to estimate $\beta(s) = \int_M G^2(x, t; y, s) \, d\mu_{(y,s)}$, for which the computation is different, due to the assymetry of the equation. As stated above, the second entries of G satisfy the conjugated equation:

$$\Delta_y G(x, t; y, s) + \partial_s G(x, t; y, s) - RG(x, t; y, s) = 0 \, .$$

Proceeding just as in the $\alpha(t)$ case, we get the following:

$$\begin{aligned}
\beta'(s) &= \int_M \left(2GG_s - RG^2\right) d\mu_{(y,s)} = 2\int_M G(-\Delta G + RG)\, d\mu_{(y,s)} - \int_M RG^2\, d\mu_{(y,s)} \\
&= -2\int_M G(\Delta G)\, d\mu_{(y,s)} + \int_M RG^2\, d\mu_{(y,s)} = 2\int_M |\nabla G|^2\, d\mu_{(y,s)} + \int_M RG^2\, d\mu_{(y,s)} \\
&\geq \int_M |\nabla G|^2\, d\mu_{(y,s)} + \int_M RG^2\, d\mu_{(y,s)}\,.
\end{aligned}$$

Hence

$$\beta'(s) \geq \int_M (|\nabla G|^2 + RG^2)\, d\mu_{(y,s)}\,.$$

But this time, by the property of the heat kernel:

$$\tilde{J}(s) := \int_M G(x,t;y,s)\, d\mu_{(y,s)} = 1\,,$$

so by applying Hölder (as for $\alpha(t)$) and relating it to the Sobolev inequality, we find:

$$\begin{aligned}
&\int_M G^2\, d\mu_{(y,s)} \leq \\
&\left[A(s)\int_M \left(|\nabla G|^2 + \frac{1}{4}RG^2\right) d\mu_{(y,s)} + B(s)\int_M G^2\, d\mu_{(y,s)}\right]^{\frac{n}{n+2}} \left(\int_M G\, d\mu_{(y,s)}\right)^{\frac{4}{n+2}} \\
&= \left[A(s)\int_M \left(|\nabla G|^2 + \frac{1}{4}RG^2\right) d\mu_{(y,s)} + B(s)\int_M G^2\, d\mu_{(y,s)}\right]^{\frac{n}{n+2}}\,.
\end{aligned}$$

Following the same steps as for $\alpha(t)$, one finds

$$\beta'(s) \geq \frac{1}{A(s)}\beta(s)^{\frac{n+2}{n}} - h(s)\beta(s)$$

($h(s)$ denotes, as before, $\frac{B(s)}{A(s)} - \frac{3}{4}\cdot\frac{1}{m_0 - c_n s}$).

The above is true for any $\tau \in [s, t)$. We will apply again the integrating factor method, with $H(\tau)$ being the same antiderivative of $h(\tau)$ as above. For $\tau \in [s, t)$, the following holds:

$$\left(e^{H(\tau)}\beta(\tau)\right)' \geq \frac{1}{A(\tau)}\left(e^{H(\tau)}\beta(\tau)\right)^{\frac{n+2}{n}} e^{-\frac{2}{n}H(\tau)} .$$

Integrating between s and t, and taking into account that

$$\lim_{\tau \nearrow t} \beta(\tau) = \int_M \lim_{\tau \nearrow t} G^2(x, t; y, \tau)\, d\mu(y, \tau) = \int_M \delta_y^2(x)\, d\mu(y, t) = 0 ,$$

we get the second desired bound:

$$\beta(s) \leq \frac{C_n e^{-H(s)}}{\left(\int_s^t \frac{e^{-\frac{2}{n}H(\tau)}}{A(\tau)}\, d\tau\right)^{n/2}} .$$

From the estimates of α and β we obtain the following:

$$\alpha\left(\frac{t}{2}\right) = \int_M G^2\left(z, \frac{t}{2}; y, 0\right)\, d\mu_{\left(z, \frac{t}{2}\right)} \leq \frac{C_n e^{H\left(\frac{t}{2}\right)}}{\left(\int_0^{t/2} \left(\frac{m_0 - c_n\tau}{m_0}\right)^{-2} \frac{e^{\frac{2}{n}H(\tau)}}{A(\tau)}\, d\tau\right)^{\frac{n}{2}}} ,$$

$$\beta\left(\frac{t}{2}\right) = \int_M G^2\left(x, t; z, \frac{t}{2}\right)\, d\mu_{\left(z, \frac{t}{2}\right)} \leq \frac{C_n e^{-H\left(\frac{t}{2}\right)}}{\left(\int_{t/2}^t \frac{e^{-\frac{2}{n}H(\tau)}}{A(\tau)}\, d\tau\right)^{n/2}} .$$

Here, we may choose $H\left(\frac{t}{2}\right) = \int_0^{t/2} \left[\frac{B(\tau)}{A(\tau)} - \frac{3}{4} \cdot \frac{1}{m_0 - c_n\tau}\right] d\tau$, since the relation is true for any antiderivative of $h(\tau) = \frac{B(\tau)}{A(\tau)} - \frac{3}{4} \cdot \frac{1}{m_0 - c_n\tau}$.

The conclusion follows from multiplying the relations above.

3.5 Special case: positive scalar curvature

In the special case when $R(x,t) > 0$, one gets that $J'(\tau) \leq 0$, which means that $J(\tau)$ is decreasing, so $J(\tau) \leq J(s) = 1$, thus leading to the differential inequality for $\alpha(t)$ to be:

$$\alpha'(t) \leq -\frac{1}{A(t)}\alpha(t)^{\frac{n+2}{n}} + \frac{B(t)}{A(t)}\alpha(t) \,.$$

And from this the bound for $\alpha(t)$ becomes:

$$\alpha(t) \leq \frac{C_n e^{H(t)}}{\left(\int\limits_s^t \frac{e^{\frac{2}{n}H(\tau)}}{A(\tau)} d\tau\right)^{\frac{n}{2}}} \,,$$

where $H(\tau)$ is the antiderivative of $\frac{B(\tau)}{A(\tau)}$ such that $H(s) \neq 0$ and $H(t) \neq 0$.

Similarly, one obtaines for $\beta(s)$:

$$\beta'(s) \geq \frac{1}{A(s)}\beta(s)^{\frac{n+2}{n}} - \frac{B(s)}{A(s)}\alpha(s) \,,$$

and from this:

$$\beta(s) \leq \frac{C_n e^{-H(s)}}{\left(\int\limits_s^t \frac{e^{-\frac{2}{n}H(\tau)}}{A(\tau)} \, d\tau\right)^{n/2}} \,,$$

where $H(\tau)$ is the same antiderivative of $\frac{B(\tau)}{A(\tau)}$ as above.

By (22), in the case of $R(x, 0) > 0$, the two functions $A(t)$ and $B(t)$ are constants, let's call them A and B. Recall that A is in fact $K(n, 2)^2$, where $K(n, 2)$ is the best constant in the Sobolev imbedding and $\epsilon > 0$.

One has that $H(t) = \frac{B}{A}t$. Using this, we obtain:

$$G(x, t; y, s) \leq \frac{C_n}{\left(\int\limits_{s}^{\frac{s+t}{2}} \frac{e^{\frac{2}{n}H(\tau)}}{A(\tau)}\, d\tau \right)^{\frac{n}{4}} \left(\int\limits_{\frac{s+t}{2}}^{t} \frac{e^{-\frac{2}{n}H(\tau)}}{A(\tau)}\, d\tau \right)^{\frac{n}{4}}} = \frac{C_n}{\left[\frac{n^2}{4B^2} \left(1 - e^{-\frac{2B}{nA}\frac{t-s}{2}}\right)^2 \right]^{\frac{n}{4}}}.$$

But, by Taylor expansion, the last expression is bounded by

$$\frac{C_n}{\left[\frac{n^2}{4B^2} \left(1 - e^{-\frac{2B}{nA}\frac{t-s}{2}}\right)^2 \right]^{\frac{n}{4}}} \leq \frac{\tilde{C}_n}{(t-s)^{\frac{n}{2}}},$$

where $\tilde{C}_n = C_n \cdot (2A)^{\frac{n}{2}} = \left(\frac{4}{n}\right)^{\frac{n}{2}} \cdot (K(n, 2))^n$.

Combining the two, the desired corollary follows.

CHAPTER 4

APPLICATION: TYPE III κ-SOLUTION TO THE RICCI FLOW

4.1 Introduction

In this chapter we will try to use the gradient estimates and the bounds for the heat kernel on a more specific type of solution to the Ricci flow. More precisely, we will analyze type III κ-solutions to the Ricci flow.

We will begin by defining the different concepts used. In [17] Hamilton introduced a classification of solutions to the Ricci flow, in terms of the maximum interval of existence (i.e. until the curvature blows up). We consider a manifold M, which is compact or complete with bounded curvature, and let $[0, T]$ be the interval of existence for the Ricci flow 3.2.1, for $T > 0$. Note that either $T = \infty$ (solution goes on forever) or $|\operatorname{Rm}|$ is unbounded as t approaces T (the curvature blows up). The starting point of the interval can be a negative number too, but for simplicity let's assume it is 0. Let $M(t) = \sup\{|\operatorname{Rm}(x, t)|\}$, where the supremum is taken over all points x of the manifold at time t.

A maximal solution is one and only one of the following three types:

- Type I: $T < \infty$ and $\sup(T - t)M(t) < \infty$.
- Type II(a): $T < \infty$ and $\sup(T - t)M(t) = \infty$.
- Type II(b): $T = \infty$ and $\sup tM(t) = \infty$.
- Type III: $T = \infty$ and $\sup tM(t) < \infty$.

We will focus on solutions of type III, hence the solution is defined on $[0, \infty)$

and at any time $t > 0$, $|\mathrm{Rm}| < \frac{c}{t}$ for some positive constant t. This implies that all curvatures will be bounded by $\frac{c}{t}$ multiplied by a dimensional constant.

Moreover, we will assume the solution is a κ-solution, or it is κ-non-collapsed on all scales. This is a concept introduced by Perelman in his proof of the Poincaré conjecture and was used to analyze the blow-ups. He proved that the rescaling limits at singularities of the Ricci flow are κ-solutions. The definition is as follows: For $x_0 \in M$, define $P(x_0, t_0, r, -r^2)$ to be the parabolic ball $\{(x,t)|d(x, x_0, t) < r, t_0 - r^2 < t < t_0\}$. The manifold is κ-non-collapsed on all scales if for any $r > 0$, if $|\mathrm{Rm}| \leq r^{-2}$ on the parabolic ball, then $|B(x_0, r, t_0)|_{t_0} \geq \kappa r^n$. Here $|B(x_0, r, t_0)|_{t_0}$ denotes the volume of the geodesic ball centered at x_0 with radius r at time t_0, which is measured with respect to the metric $g(t_0)$.

We will analyze dilation limits of type III solutions. We let $(M, g(t))$ for $t \in [0, \infty)$ be a non-flat, type III κ-solutions to the Ricci flow. Consider a sequence of points $\{x_k\} \subset M$, a sequence of times $t_k \to \infty$ and a sequence of rescaled metrics $g_k(x,s) := \frac{1}{t_k} g(x, st_k)$ around x_k ($s \in [1,3]$ - the choice of the compact interval is purely random, it can be any compact interval with positive endpoints). Note that any g_k is again a Ricci flow solution, in the "new" time variable s.

It is known, by a compactness theorem of Hamilton [20] that such a sequence will converge in the C^∞_{loc} topology to a manifold, which is also a Ricci flow solution (in the time variable s), $(M_\infty, g_\infty(x,s))$. In order for the theorem to hold, two conditions need to be satisfied:

- the sectional curvature to be uniformly bounded by above - this is true, since $|\mathrm{Rm}| < \frac{c}{t}$;
- the injectivity radius to be bounded from below - this is again true, since

we deal with κ-solutions.

It is interesting to understand what kind of Ricci flow solution is $(M_\infty, g_\infty(x,s))$. We believe that this should be a non-flat gradient expanding Ricci soliton. The reason for that is the following result by Cao and Zhang [11]:

Theorem 25 (X. Cao and Q. S. Zhang [11]) *Let $(M, g(t))$, $t \in (-\infty, 0]$ be a non-flat, type I κ-solution to the Ricci flow. Then there is a sequence of points $\{q_k\} \subset M$, a sequence of times $t_k \to -\infty$ and a sequence of rescaled metrics*

$$g_k(x,s) = |t_k|^{-1} g(x, t_k - s|t_k|)$$

around q_k such that (M, g_k, q_k) converge to a non-flat gradient shrinking Ricci soliton in C^∞_{loc} topology.

Type I solution can be regarded as a mirrored type III solution (in type I $|\mathrm{Rm}| < \frac{c}{|-t|}$, for $t \in (-\infty, 0)$, while in type III $|\mathrm{Rm}| < \frac{c}{t}$, for $t \in (0, \infty)$), hence one may expect a similar approach between the two.

The essential step of the proof of theorem 25 was the following heat kernel estimate

$$\frac{a}{t^{n/2}} \leq G(x_0, 0; x_0, -t) \leq \frac{b}{t^{n/2}}$$

which follows from a Perelman type Harnack inequality for the conjugate heat equation $\Delta u + u_t - Ru = 0$. Note that a, b are positive numbers depending only on the dimension of the manifold, κ and the bound for the curvature tensor.

Hence in order to attack the problem, we need to prove a Harnack inequality for the heat equation under type III Ricci flow and to find bounds on the heat kernel.

4.2 Harnack inequality

By using theorem 16 and replacing $k_1 = k_2 = \frac{c}{t}$ for $c > 0$ we obtain the following theorem:

Theorem 26 *Let $(M, g(x,t))_{t\in[0,\infty)}$ be a complete type III maximal solution to the Ricci flow* (1.2.1) *(which implies $-\frac{c}{t}g(x,t) \le \mathrm{Ric}(x,t) \le \frac{c}{t}g(x,t)$ for some $c > 0$ and all $(x,t) \in B_{\rho,T}$). Consider a smooth positive function $u : M \times [0,\infty) \to \mathbb{R}$ solving the heat equation $\Delta u - u_t = 0$. There exists a constant C, that depends only on the dimension of M and c, which satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \le \frac{C\alpha^3}{\alpha - 1}\left(\frac{\alpha}{\rho^2} + \frac{1}{t}\right) \tag{4.2.1}$$

for all $\alpha > 1$ and all $(x,t) \in B_{\frac{\rho}{2},T}$ with $t \neq 0$.

By letting $\rho \to \infty$ in the above theorem, the following corollary follows:

Corollary 27 *Let $(M, g(x,t))_{t\in[0,\infty)}$ be a complete type III maximal solution to the Ricci flow and assume that it has no boundary. Then there exists a constant C, that depends only on the dimension of M and c, which satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \le \frac{C\alpha^3}{t(\alpha - 1)} \tag{4.2.2}$$

for all $\alpha > 1$.

For u being a solution to the heat equation (2.2.2), define h to be the function such that

$$u = (4\pi t)^{-\frac{n}{2}} e^h\,,$$

for any $t > 0$. From this it follows that:

$$\begin{aligned}
h &= f + \frac{n}{2}\log(4\pi t)\,, \\
h_t &= f_t + \frac{n}{2t}\,, \\
\Delta h &= \Delta f\,, \\
|\nabla h|^2 &= |\nabla f|^2\,, \\
\Delta h &= |\nabla h|^2 + h_t - \frac{n}{2t}\,.
\end{aligned}$$

By corollary 27 h satisfies the following:

$$|\nabla h|^2 - \alpha h_t \leq \frac{C\alpha^3}{t(\alpha-1)} - \frac{n\alpha}{2t}\,. \tag{4.2.3}$$

We can now obtain a Perelman-type Harnack inequality:

Corollary 28 *For any smooth curve $\gamma(t)$ in M the following holds*

$$-\frac{d}{dt}h(\gamma(t),t) \leq \frac{1}{2}|\dot{\gamma}|^2 - \frac{\hat{C}}{2t}\,.$$

PROOF. Taking $\alpha = 2$ in (4.2.3) we obtain

$$h_t - \frac{1}{2}|\nabla h|^2 \geq \frac{\hat{C}}{2t}\,.$$

For any curve

$$-\frac{d}{dt}h(\gamma(t),t) \leq -h_t + \frac{1}{2}|\nabla h|^2 + \frac{1}{2}|\dot{\gamma}|^2\,.$$

Adding the two we obtain the conclusion.

BIBLIOGRAPHY

[1] M. Arnaudon, K.A. Coulibaly, and A. Thalmaier. Brownian motion with respect to a metric depending in time: definition, existence and applications to ricci flow. *C. R. Math. Acad. Sci. Paris*, 346:773–778, 2008.

[2] D. G. Aronson. Bound for the fundamental solution of a parabolic equation. *Bull. Am. Math. Soc.*, 73:890–896, 1967.

[3] D. G. Aronson and P. Bénilan. Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^n$. *C. R. Acad. Sci. Paris Sér. A-B*, 288:A103–A105, 1979.

[4] T. Aubin. Problèmes isopérimetriques et éspaces de sobolev. *J. Differential Geom.*, 11:573–598, 1976.

[5] M. Bailesteanu. Bounds on the heat kernel under the ricci flow. *Proc. Amer. Math. Soc. (to appear)*, 2011.

[6] M. Bailesteanu, X. Cao, and A. Pulemotov. Gradient estimates for the heat equation under the ricci flow. *J. Funct. Anal.*, 258(10):3517–3542, 2010.

[7] D. Bakry and Z.M. Qian. Harnack inequalities on a manifold with positive or negative ricci curvature. *Rev. Mat. Iberoamericana*, 15:143–179, 1999.

[8] S. Brendle and R. Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22:287–307, 2009.

[9] X. Cao. Differential harnack estimates for backward heat equations with potentials under the ricci flow. *J. Funct. Anal.*, 255:1024–1038, 2008.

[10] X. Cao and R. Hamilton. Differential harnack estimates for time-dependent heat equations with potentials. *Geom. Funct. Anal.*, 2009.

[11] X. Cao and Q. Zhang. The conjugate heat equation and ancient solutions of the ricci flow. *arXiv:math/1006.0540v1*, 2010.

[12] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni. *The Ricci flow: techniques and applications. Part I. Geometric aspects*. American Mathematical Society, Providence, RI, 2007.

[13] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni. *The Ricci flow: techniques and applications. Part II. Analytic aspects*. American Mathematical Society, Providence, RI, 2008.

[14] B. Chow and R. Hamilton. Constrained and linear harnack inequalities for parabolic equations. *Invent. Math.*, 129(2):213–238, 1997.

[15] B. Chow, P. Lu, and L. Ni. *Hamilton's Ricci flow*. American Mathematical Society, Providence, RI; Science Press, New York, 2006.

[16] C.M. Guenther. The fundamental solution on manifolds with time-dependent metrics. *J. Geom. Anal.*, 12:425–436, 2002.

[17] R.S. Hamilton. The formation of singularities in the ricci flow. *Surveys in Differential Geometry*.

[18] R.S. Hamilton. Three-manifolds with positive ricci curvature. *J. Differential. Geom.*, 17:255–306, 1982.

[19] R.S. Hamilton. A matrix harnack estimate for the heat equation. *Comm. Anal. Geom.*, 1:113–126, 1993.

[20] R.S. Hamilton. A compactness property for solutions of the ricci flow. *American J. of Math.*, 117:545–572, 1995.

[21] R.S. Hamilton. The harnack estimate for the ricci flow. *J. Differential. Geom.*, 41:215–226, 1995.

[22] E. Hebey and M. Vaugon. Meilleures constantes dans le théorème d'inclusion de sobolev. *Ann. Inst. H. Poincaré - Anal. non-linéaire*, 13:57–93, 1996.

[23] S.-Y. Hsu. Some results for the perelman lyh-type inequality. *arXiv:0801.3506v2 [math.DG]*, 2008.

[24] B. Kleiner and J. Lott. *Notes on Perelman's Papers*. arXiv:math/0605667, 2006.

[25] P. Li and S.-T. Yau. On the parabolic kernel of the schroedinger operator. *Acta Math.*, 156:153–201, 1986.

[26] J. Morgan and G. Tian. *Ricci flow and the Poincaré conjecture*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[27] R. Mueller. *Differential Harnack inequalities and the Ricci flow*. EMS Series of lectures in mathematics, EMS Zurich, Switzerland, 2006.

[28] L. Ni. Ricci flow and nonnegativity of sectional curvature. *Math. Res. Lett.*, 11:883–904, 2004.

[29] G. Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv:math/0211159v1*, 2002.

[30] G. Perelman. Ricci flow with surgery on three-manifolds. *arXiv:math/0303109v1*, 2003.

[31] L. Saloff-Coste. *Aspects of Sobolev-Type Inequalities*. London Mathematical Society Lecture Note Series; Cambridge University Press, 2001.

[32] R. Schoen and S.-T. Yau. *Lectures on differential geometry*. International Press, Cambridge, MA, 1994.

[33] P. Souplet and Q. Zhang. Sharp gradient estimate and yau's liouville theorem for the heat equation on noncompact manifolds. *Bull. London Math. Soc.*, 38:1045–1053, 2006.

[34] J. Wang. Global heat kernel estimates. *Pacific J. Math.*, 178:377–398, 1997.

[35] Q. Zhang. Some gradient estimates for the heat equation on domains and for an equation by perelman. *Int. Math. Res. Not.*, 2006:1–39, 2006.

[36] Q. Zhang. *Sobolev inequalities, heat kernels under Ricci flow and the Poincaré conjecture*. Chapman Hall/CRC Press, Taylor and Francis group, Boca Raton, FL, 2011.